

ON THE FIBERS OF THE CANNON-THURSTON MAP FOR FREE-BY-CYCLIC GROUPS

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ABSTRACT. Let $\Phi \in \text{Aut}(F_N)$ be an atoroidal iwip automorphism of a free group F_N , $N \geq 3$ and let $M_\Phi = F_N \rtimes_\Phi \langle t \rangle$ be the mapping torus group of Φ . The group M_Φ is Gromov-hyperbolic and it follows from the work of Mitra that the inclusion $\iota : F_N \rightarrow M_\Phi$ extends to a continuous surjective map $\hat{\iota} : \partial F_N \rightarrow \partial M_\Phi$, called the *Cannon-Thurston map*. We study the fibers of the map $\hat{\iota}$.

We prove that for any Φ as above, the map $\hat{\iota}$ is finite-to-one and that the preimage of every point of ∂M_Φ has cardinality $\leq 2N$.

We also prove that every point $S \in \partial M_\Phi$ with ≥ 3 preimages in ∂F_N is rational and has the form $(xt^m)^\infty$ where $x \in F_N, m \neq 0$, and that there are at most $4N - 5$ F_N -orbits of such points in ∂M_Φ (for the translation action of F_N on ∂M_Φ).

We show that, by contrast, for $k = 1, 2$ there are uncountably many points $S \in \partial M_\Phi$ with exactly k preimages in ∂F_N .

1. INTRODUCTION

Recall that if G is a word-hyperbolic group, it possesses a canonical (in the topological sense) *hyperbolic compactification* $\overline{G} = G \sqcup \partial G$, where ∂G is the *hyperbolic boundary* of G . The action of G on itself by left translation extends to a left action of G on ∂G by homeomorphisms. We refer the reader to [26, 31] for the background information on the boundaries of hyperbolic groups.

The notion of a Cannon-Thurston map goes back to the 1984 preprint of Cannon and Thurston that was eventually published in 2007 [12]. They consider a closed hyperbolic 3-manifold M fibering over a circle with the fiber being a closed hyperbolic surface S . Then the inclusion $S \subseteq M$ lifts to the map between their universal covers $i : \mathbb{H}^2 \rightarrow \mathbb{H}^3$, where $\mathbb{H}^2 = \tilde{S}$ and $\mathbb{H}^3 = \tilde{M}$. Cannon and Thurston prove in [12] that the map i extends to a continuous $\pi_1(S)$ -equivariant map between the hyperbolic boundaries $\hat{\iota} : \mathbb{S}^1 \rightarrow \mathbb{S}^2$, where $\mathbb{S}^1 = \partial \mathbb{H}^2$ and $\mathbb{S}^2 = \partial \mathbb{H}^3$. The map $\hat{\iota}$ is necessarily surjective, and so, being a continuous map from the circle to the two-sphere, it gives a space-filling curve. Moreover, the map $\hat{\iota}$ is finite-to-one, and the full preimage of every point of \mathbb{S}^2 has cardinality at most $4g - 2$, where g is the genus of the fiber S . In group-theoretic terms, in this example we have an inclusion $H \leq G$, where $H = \pi_1(S)$ and $G = \pi_1(M)$ both word-hyperbolic, and the Cannon-Thurston theorem says that this inclusion extends to a continuous H -equivariant map between their hyperbolic boundaries $\partial H \approx \mathbb{S}^1$ and $\partial G \approx \mathbb{S}^2$. This fact led to the following definition (see Definition 5.1 below for a more precise statement):

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If G is a word-hyperbolic group and $H \leq G$ be a word-hyperbolic subgroup, and if the inclusion $\iota : H \rightarrow G$ extends to a continuous map $\hat{\iota} : \partial H \rightarrow \partial G$, the map $\hat{\iota}$ is called the *Cannon-Thurston map*. In particular, if the Cannon-Thurston map $\hat{\iota} : \partial H \rightarrow \partial G$ exists, then for any sequence $h_n \in H \cup \partial H$ converging to some $X \in \partial H$ in the topology of $H \cup \partial H$, we have $\lim_{n \rightarrow \infty} h_n = \hat{\iota}(X)$ in $G \cup \partial G$.

Analogues and relatives of the Cannon-Thurston map have also been investigated in other contexts arising in the study of hyperbolic 3-manifolds and mapping class groups (e.g. see [9, 10, 37, 38, 42, 45, 50]), of relatively hyperbolic groups [22, 23, 24, 25, 51], and of the dynamics of complex polynomials (e.g. see [30, 44, 43, 55]).

As explained in Remark 5.2, the definition of $\hat{\iota}$ implies that if $H \leq G$ are hyperbolic groups and the Cannon-Thurston map $\hat{\iota}$ exists then $\hat{\iota}(\partial H)$ is equal to the *limit set* ΛH of H in ∂G , that is, the set of all points in ∂G that are limits (in $\overline{G} = G \cup \partial G$) of sequences of elements of H . It is well-known that if $H \leq G$ is a quasiconvex (i.e. quasi-isometrically embedded) subgroup of a word-hyperbolic group G then H is word-hyperbolic and the inclusion $H \leq G$ extends to a continuous topological embedding $\partial H \rightarrow \partial G$. Thus in this case the Cannon-Thurston map exists and, moreover, is injective.

Surprisingly, it turns out that the Cannon-Thurston map exists in many situations where $H \leq G$ is not quasiconvex, as shown by the work of Mitra in 1990s [46, 47, 48, 49].

In particular, a result of Mitra [47] states that whenever

$$(\spadesuit) \quad 1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

is a short exact sequence of word-hyperbolic groups, then the inclusion $H \leq G$ extends to a continuous Cannon-Thurston map $\hat{\iota} : \partial H \rightarrow \partial G$. It is well-known [1] that in this situation if H and Q are infinite then $H \leq G$ is not quasiconvex. Also, if H in (\spadesuit) is infinite then the limit set of H in ∂G is equal to ∂G [36] and therefore the map $\hat{\iota} : \partial H \rightarrow \partial G$ is onto (see Remark 5.3 below). This result of Mitra generalizes the original theorem of Cannon and Thurston mentioned above since in that context one has a short exact sequence $1 \rightarrow \pi_1(S) \rightarrow \pi_1(M) \rightarrow \mathbb{Z} \rightarrow 1$.

Until recently it has been unknown if there are any inclusions $H \leq G$ (with H and G word-hyperbolic) where the Cannon-Thurston map does not exist [48, 31]. A surprising new result of Baker and Riley [2] constructs the first example of such an inclusion (with $H = F_3$) where the Cannon-Thurston map does not exist. Their results were subsequently further extended by Matsuda and Oguni [41].

The result of Mitra, mentioned above, applies, in particular, to word-hyperbolic free-by-cyclic groups. Recall that if $\Phi \in \text{Aut}(F_N)$ is an automorphism of F_N , then the *mapping torus* group of Φ is

$$M_\Phi = F_N \rtimes_\Phi \langle t \rangle = \langle F_N, t | tht^{-1} = \Phi(h), h \in F_N \rangle.$$

An automorphism Φ of F_N is called *hyperbolic* if the group M_Φ is word-hyperbolic. It follows from the Bestvina-Feighn Combination Theorem [4] and a result of Brinkmann [11] that $\Phi \in \text{Out}(F_N)$ is hyperbolic if and only if $N \geq 3$ and Φ is *atoroidal*, that is, does not have any nontrivial periodic conjugacy classes in F_N (which is also equivalent to the condition that M_Φ does not contain any $\mathbb{Z} \times \mathbb{Z}$ -subgroups). An element $\varphi \in \text{Out}(F_N)$ is called *hyperbolic* if some (equivalently, any) representative $\Phi \in \text{Aut}(F_N)$ of φ is hyperbolic. Note that replacing t by $t_1 = ut$ in the presentation of M_Φ above, where $u \in F_N$ is arbitrary, rewrites the relation $tht^{-1} = \Phi(h)$ into $t_1 ht = u\Phi(h)u^{-1}$. Thus M_Φ and the inclusion $F_N \leq M_\Phi$

depend only on the outer automorphism class φ of Φ . So, if $\Phi \in \text{Aut}(F_N)$ is a hyperbolic automorphism then we have a short exact sequence

$$1 \rightarrow F_N \rightarrow M_\Phi \rightarrow \langle t \rangle \rightarrow 1$$

of three word-hyperbolic groups, and hence, by Mitra's theorem, there does exist a continuous F_N -equivariant Cannon-Thurston map $\hat{\iota} : \partial F_N \rightarrow \partial M_\Phi$. In this case for a point $S \in \partial M_\Phi$ let the *degree of S* , denoted $\deg(S)$, be the cardinality of the set $(\hat{\iota})^{-1}(S)$. Since $\hat{\iota}$ is surjective, for every $S \in \partial F_N$ we have $\deg(S) \geq 1$.

By now the properties of the Cannon-Thurston map in the original context of [12] of a closed hyperbolic 3-manifold fibering over a circle are very well understood. By contrast, apart from its existence, little has been known about the specific properties of the Cannon-Thurston map for mapping torus groups of hyperbolic automorphisms of free groups. The most typical type of hyperbolic automorphisms of free groups are so-called iwip or “fully irreducible” hyperbolic automorphisms. Recall that an element $\varphi \in \text{Out}(F_N)$ is said to be *irreducible with irreducible powers* (*iwip*, for short), or *fully irreducible* if no positive power of φ preserves the conjugacy class of a proper free factor of F_N . Bestvina and Handel proved [3] that if an iwip $\varphi \in \text{Out}(F_N)$ fails to be atoroidal (i.e., in view of the above discussion, fails to be hyperbolic) then φ is induced by a homeomorphism of a compact connected surface with a single boundary component. Thus, for $N \geq 3$, “most” iwips are atoroidal. By contrast, it is easy to see that for $N = 2$ there are no atoroidal elements in $\text{Out}(F_2)$. Moreover, in a sense made precise by Rivin [54], for $N \geq 3$ a “random” element of $\text{Out}(F_N)$ is a hyperbolic iwip.

We can now state the first result of this paper:

Theorem A. *Let $N \geq 3$ and let $\varphi \in \text{Out}(F_N)$ be a hyperbolic iwip. Let $\Phi \in \text{Aut}(F_N)$ be a representative of φ and let $M_\Phi = F_N \rtimes_\Phi \mathbb{Z}$ be the mapping torus group of Φ . Let $\hat{\iota} : \partial F_N \rightarrow \partial M_\Phi$ be the Cannon-Thurston map.*

Then for every $S \in \partial M_\Phi$ we have $\deg(S) \leq 2N$.

Moreover, as we note in Remark 7.2, the $2N$ bound in Theorem A is sharp, that is, there exist φ, Φ as in Theorem A such that for some $S \in \partial M_\Phi$ we have $\deg(S) = 2N$.

In [46] Mitra gave a description of the fibers of the Cannon-Thurston map $\hat{\iota} : \partial H \rightarrow \partial G$ for any short exact sequence (\spadesuit) of three hyperbolic groups $1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$. This description is given in terms of “ending laminations” $\Lambda_z, z \in \partial Q$, where $\Lambda_z \subseteq \partial^2 H = \{(X, Y) \in \partial H \times \partial H : X \neq Y\}$. Given a hyperbolic iwip $\varphi \in \text{Out}(F_N)$, there are several naturally associated to φ “laminations” $\subseteq \partial^2 F_N$ that arose in the study of $\text{Out}(F_N)$; see Section 3. The laminations $L_{BFH}(\varphi^{\pm 1}) \subseteq \partial^2 F_N$ were introduced by Bestvina, Feighn and Handel in [6] and are defined in terms of train tracks representing φ . The laminations $L(T_\pm(\varphi))$ of the trees $T_\pm(\varphi) \in \overline{\text{CV}}_N$ of φ are special cases of the general notion of a “dual” or “zero” lamination $L(T)$ for $T \in \overline{\text{CV}}_N$ introduced in [17]. Here $[T_\pm(\varphi)] \in \overline{\text{CV}}_N$ are the attracting/repelling fixed points for the (right) action of φ on the compactified Outer space $\overline{\text{CV}}_N$. In our earlier work [35] we showed that for a hyperbolic iwip $\varphi \in \text{Out}(F_N)$ we have $L(T_-(\varphi)) = \text{diag}(L_{BFH}(\varphi))$, the “diagonal extension” of $L_{BFH}(\varphi)$. See Section 3 below for precise definition of diagonal extension. The first step in the proof of Theorem A is establishing, using our results from [35], Proposition 6.4. This proposition relates Mitra’s “ending laminations”, for the short exact sequence corresponding to the mapping torus group of a hyperbolic iwip

$\varphi \in \text{Out}(F_N)$, to the laminations $L(T_{\pm}(\varphi))$. Then, by Mitra's results from [46], Proposition 6.4 implies Corollary 6.5 which states that for the Cannon-Thurston map $\widehat{\iota}: \partial F_N \rightarrow \partial M_{\Phi}$ and for distinct $X, Y \in \partial F_N$ we have $\widehat{\iota}(X) = \widehat{\iota}(Y)$ if and only if $(X, Y) \in L(T_{-}(\varphi)) \cup L(T_{+}(\varphi))$. Corollary 6.5 is a key fact for our analysis of the fibers of the Cannon-Thurston map.

We then use a description, due to Coulbois, Hilion and Lustig in [17], of the dual lamination $L(T)$, where $T \in \overline{\text{cv}}_N$ is a tree with dense F_N -orbits (e.g. $T = T_{\pm}(\varphi)$) in terms of the so-called \mathcal{Q} -map. We combine this description of $L(T_{\pm}(\varphi))$ with the results of the “index” theory for points in $\overline{\text{cv}}_N$ and elements of $\text{Out}(F_N)$, particularly the known bounds for the \mathcal{Q} -index of $T_{\pm}(\varphi)$, to derive the conclusion of Theorem A.

After the proof of Theorem A we undertake a more detailed study of the fibers of the Cannon-Thurston map. We say that $S \in \partial M_{\Phi}$ is *regular* if $\deg(S) = 1$, that S is *singular* if $\deg(S) \geq 3$ and that S is *semi-singular* if $\deg(S) = 2$. It is straightforward to show that $\deg(S) = \deg(gS)$ for any $S \in \partial M_{\Phi}$ and $g \in M_{\Phi}$. The group $G = M_{\Phi}$ acts on ∂M_{Φ} by translations, and hence so thus $F_N \leq M_{\Phi}$. When referring to G -orbits or F_N -orbits of points in ∂G , we will mean these translation actions.

We next obtain the following result giving fairly precise information about singular points in ∂M_{Φ} :

Theorem B. *Let $N \geq 3$, $\Phi \in \text{Aut}(F_N)$ be a hyperbolic iwip and let M_{Φ} be its mapping torus group. Then:*

- (1) *Every singular point $S \in \partial M_{\Phi}$ is rational and has the form $S = (xt^m)^{\infty}$ for some $x \in F_N$ and $m \neq 0$.*
- (2) *The number σ of F_N -orbits of singular points in ∂M_{Φ} is finite and satisfies $2 \leq \sigma \leq 4N - 5$.*
- (3) *We have*

$$\sum_{[S]} (\deg(S) - 2) \leq 4N - 5$$

where the sum is taken over all F_N -orbits $[S]$ of singular points in ∂M_{Φ} .

Theorem B implies that for every singular $S \in \partial M_{\Phi}$ there exists a unique $g \in M_{\Phi}$ such that g is not a proper power and such that $g^{\infty} = S$; moreover, there are $\leq 4N - 5$ conjugacy classes of $g \in G$ with these properties.

We next summarize, in a simplified form, the remaining results (obtained in Section 8) about fibers of $\widehat{\iota}$ for M_{Φ} .

Theorem C. *Let $N \geq 3$, $\Phi \in \text{Aut}(F_N)$ be a hyperbolic iwip and let M_{Φ} be its mapping torus group. Then the following hold:*

- (1) *For any $S \in \partial M_{\Phi}$ and any $g \in M_{\Phi}$ we have $\deg(S) = \deg(gS)$.*
- (2) *Let $g = xt^m \in M_{\Phi}$ where $x \in F_N$ and $m \neq 0$. Then*

$$\deg(g^{\infty}) + \deg(g^{-\infty}) \leq 4N - 1.$$

- (3) *If $x \in F_N, x \neq 1$ then the point $x^{\infty} \in \partial M_{\Phi}$ is regular.*
- (4) *There are uncountably many M_{Φ} -orbits of regular points in ∂M_{Φ} . (Since there are only countably many rational points in ∂M_{Φ} , this also implies that there are uncountably many M_{Φ} -orbits of irrational regular points in ∂M_{Φ} .)*

- (5) *There are uncountably many M_Φ -orbits of semi-singular points in ∂M_Φ . (Again, this also implies that there are uncountably many M_Φ -orbits of irrational semi-singular points in ∂M_Φ).*

The paper is organized as follows. We give preliminary background information in Section 2. In Section 3 we discuss the general notion of an algebraic lamination on a free group, and discuss the attracting/repelling trees $T_\pm(\varphi)$ for an iwip φ and several algebraic laminations naturally associated to iwip elements of $\text{Out}(F_N)$. Section 4 provides an overview of the relevant index theory, including the notions of geometric index and \mathcal{Q} -index for trees, and the notion of index for free group automorphisms. In Section 5 we give a precise definition of the Cannon-Thurston map and note some of its basic properties. In Section 6 we discuss Mitra's "ending laminations" and obtain Proposition 6.4 and Corollary 6.5 which, for a hyperbolic iwip $\Phi \in \text{Aut}(F_N)$ characterize the fibers of the Cannon-Thurston map $\hat{\iota}: \partial F_N \rightarrow \partial M_\Phi$ in terms of the dual laminations of $T_\pm(\varphi)$. In Section 7 we obtain Theorem A. In Section 8 we obtain (more detailed versions of) Theorem B and Theorem C. In Section 9 we discuss some open problems.

The paper uses a significant amount of varied technology and for that reason the background sections necessarily take a fair amount of space in the paper. An experienced reader, well-familiar with train-tracks, algebraic laminations, dual lamination of a point of $\overline{\text{cv}}_N$, attracting/repelling trees of iwips, the map \mathcal{Q} and the index theory for $\text{Out}(F_N)$ and $\overline{\text{cv}}_N$, may want to jump directly to Section 5 and Section 6 and only refer to the background sections as needed. However, readers who are not well familiar with the above mentioned topics are well advised to read through (at least some of) the background sections first.

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2. OUTER SPACE

2.1. Conventions regarding graphs. By a *graph* Δ we mean a 1-dimensional CW-complex. We refer to the 0-cells as *vertices* of Δ , and the set of all vertices of Γ is denoted $V\Delta$. Since every closed 1-cell is homeomorphic to either $[0, 1]$ or \mathbb{S}^1 , it is an orientable 1-manifold admitting exactly two orientations. An *edge* of Δ is a closed 1-cell with a choice of an orientation on it. We also sometimes refer to open 1-cells of Δ as *open edges*. The set of all edges of Δ is denoted $E\Delta$. If $e \in E\Delta$, changing the orientation on e to the opposite one produces another edge of Δ denoted by e^{-1} . By construction $e \neq e^{-1}$ and $(e^{-1})^{-1} = e$. For every edge $e \in E\Delta$ there are obviously defined *origin* $o(e) \in V(\Delta)$ and *terminus* $t(e) \in V(\Delta)$ (note that we need to use the orientation on e to distinguish between $o(e)$ and $t(e)$ for a non-loop edge). By construction, we have $o(e) = t(e^{-1})$ and $t(e) = o(e^{-1})$. An *edge-path* of length $n \geq 1$ in Δ is a sequence $\gamma = e_1, \dots, e_n$ of edges of Δ such that $t(e_i) = o(e_{i+1})$ for $1 \leq i < n$. We denote $n = |\gamma|$ and put $o(\gamma) := o(e_1)$, $t(\gamma) := t(e_n)$. For any vertex $v \in V\Delta$ we also regard $\gamma = v$ as an edge-path of length $|\gamma| = 0$ with $o(\gamma) = t(\gamma) = v$. For $v \in V\Delta$ the *valence* $\text{val}_\Delta(v)$ of v in Δ is the number of all $e \in E\Delta$ with $o(e) = v$. An edge-path γ is *reduced* or *tight* if it does not contain a subpath of the form e, e^{-1} , where $e \in E\Delta$. An edge-path $\gamma = e_1, \dots, e_n$ (where $n \geq 1$) is *cyclically reduced* if $o(\gamma) = t(\gamma)$, the path γ is reduced and $e_n \neq e_1^{-1}$. We also regard edge-paths of length 0 as cyclically reduced

paths. Note that this definition implies that if γ is a cyclically reduced edge-path then γ is closed, that is $o(\gamma) = t(\gamma)$.

If γ is an edge-path in Δ , we denote the reduced (i.e. tightened relative end-points) form of γ by $[\gamma]$. Thus $||[\gamma]|| \leq |\gamma|$ and $o(\gamma) = o([\gamma])$, $t(\gamma) = t([\gamma])$. If γ and γ' are edge-paths such that $t(\gamma) = o(\gamma')$, we denote by $\gamma\gamma'$ the path obtained by concatenating γ and γ' , without tightening. Thus $|\gamma\gamma'| = |\gamma| + |\gamma'|$.

If Δ_1, Δ_2 are graphs, a *graph-map* from Δ_1 to Δ_2 is a continuous function $f : \Delta_1 \rightarrow \Delta_2$ such that $f(V\Delta_1) \subseteq V\Delta_2$ and such that for every $e \in E\Delta_1$ $f(e)$ is a reduced edge-path in Δ_2 with $|f(e)| \geq 1$ (and hence, by continuity, $o(f(e)) = f(o(e))$ and $t(f(e)) = t(f(e))$); we also require that after subdividing each $e \in E\Delta_1$ in $n = |f(e)|$ edges, f maps the interior of every subdivision edge homeomorphically to an open edge of Δ_2 .

Let $N \geq 2$. A *marking* on F_N is an isomorphism $\alpha : F_N \rightarrow \pi_1(\Gamma, v_0)$, where Γ is a finite connected graph without valence-1 and valence-2 vertices, and where $v_0 \in V\Gamma$. The base-vertex v_0 will almost never be relevant when markings are discussed, and therefore we will usually suppress the mention of v_0 .

Convention 2.1. Let $\alpha : F_N \rightarrow \pi_1(\Gamma)$ be a marking. When endowed with the simplicial metric (where every edge has length 1), $\tilde{\Gamma}$ becomes a 0-hyperbolic metric space. Moreover, α induces an F_N -equivariant quasi-isometry $F_N \rightarrow \tilde{\Gamma}$ and hence it induces an F_N -equivariant homeomorphism $\partial F_N \rightarrow \partial \tilde{\Gamma}$. We will often use this homeomorphism to identify ∂F_N and $\partial \tilde{\Gamma}$ and write $\partial F_N = \partial \tilde{\Gamma}$.

2.2. Outer space. We give here only a brief overview of basic facts related to Outer space. We refer the reader to [18, 27, 28, 56] for more detailed background information.

Let $N \geq 2$. The *unprojectivized Outer space* cv_N consists of all minimal free and discrete isometric actions on F_N on \mathbb{R} -trees (where two such actions are considered equal if there exists an F_N -equivariant isometry between the corresponding trees). There are several different topologies on cv_N that are known to coincide, in particular the equivariant Gromov-Hausdorff convergence topology and the so-called *axis* or *length function* topology. Every $T \in cv_N$ is uniquely determined by its *translation length function* $||\cdot||_T : F_N \rightarrow \mathbb{R}$, where $||g||_T$ is the translation length of g on T . Two trees $T_1, T_2 \in cv_N$ are close if the functions $||\cdot||_{T_1}$ and $||\cdot||_{T_2}$ are close pointwise on a large ball in F_N . The closure \overline{cv}_N of cv_N in either of these two topologies is well-understood and known to consist precisely of all the so-called *very small* minimal isometric actions of F_N on \mathbb{R} -trees, see [5] and [13]. The automorphism group $\text{Aut}(F_N)$ has a natural continuous right action on \overline{cv}_N (that leaves cv_N invariant) given at the level of length functions as follows: for $T \in cv_N$ and $\Phi \in \text{Aut}(F_N)$ we have $||g||_{T\Phi} = ||\Phi(g)||_T$, where $g \in F_N$. In terms of tree actions, $T\Phi$ is equal to T as a metric space, but the action of F_N is modified as: $g \cdot_T x = \Phi(g) \cdot_T x$ where $x \in T$, $g \in F_N$ are arbitrary. The subgroup $\text{Inn}(F_N) \leq \text{Aut}(F_N)$ is contained in the kernel of this action, and hence we also get a natural action of $\text{Out}(F_N)$ on \overline{cv}_N as well. For $\varphi \in \text{Out}(F_N)$ and $T \in \overline{cv}_N$ put $T\varphi := T\Phi$ where $\Phi \in \text{Aut}(F_N)$ is any representative of the outer automorphism φ in $\text{Aut}(F_N)$. The *projectivized Outer space* $\overline{CV}_N = \mathbb{P}\overline{cv}_N$ is defined as the quotient cv_N / \sim where for $T_1 \sim T_2$ whenever $T_2 = cT_1$ for some $c > 0$. One similarly defines the projectivization $\overline{CV}_N = \mathbb{P}\overline{cv}_N$ of $\overline{cv}(F_N)$ as $\overline{cv}(F_N) / \sim$ where \sim is the same as above. The space \overline{CV}_N is compact and contains CV_N as a dense $\text{Out}(F_N)$ -invariant subset. The compactification \overline{CV}_N of

\overline{CV}_N is a free group analog of the Thurston compactification of the Teichmüller space. For $T \in \overline{CV}_N$ its \sim -equivalence class is denoted by $[T]$, so that $[T]$ is the image of T in \overline{CV}_N .

2.3. Train tracks. Let $\varphi \in \text{Out}(F_N)$. A *topological representative* of φ is given by a marking $\alpha : F_N \rightarrow \pi_1(\Gamma, v_0)$ and a graph map $f : \Gamma \rightarrow \Gamma$ such that:

- (1) The map f is a homotopy equivalence.
- (2) For some (equivalently any) edge-path β from v_0 to $f(v_0)$ in Γ the map

$$\alpha^{-1} \circ \theta \circ \alpha : F_N \rightarrow F_N$$

is an automorphism of F_N whose outer automorphism class is φ . Here $\theta : \pi_1(\Gamma, v_0) \rightarrow \pi_1(\Gamma, v_0)$ is the map given by $[\gamma] \mapsto [\beta f(\gamma) \beta^{-1}]$ for every closed edge-path γ from v_0 to v_0 in Γ .

Definition 2.2 (Train track). Let $\varphi \in \text{Out}(F_N)$. A topological representative $f : \Gamma \rightarrow \Gamma$ of φ , with associated marking $\alpha : F_N \rightarrow \pi_1(\Gamma)$, is called a *train track representative* of φ if for every edge $e \in E\Gamma$ and every $n \geq 1$ the edge-path $f^n(e)$ is reduced.

3. ALGEBRAIC LAMINATIONS

3.1. General notion of an algebraic lamination. Let $N \geq 2$. Put

$$\partial^2 F_N := \{(X, Y) : X, Y \in \partial F_N, \text{ and } X \neq Y\}.$$

Thus $\partial^2 F_N \subseteq \partial F_N \times \partial F_N$ and we endow $\partial^2 F_N$ with induced topology from $\partial F_N \times \partial F_N$. Note that the left translation action of F_N on ∂F_N induces a natural diagonal action of F_N on $\partial^2 F_N$ by homeomorphisms: for $(X, Y) \in \partial^2 F_N$ and $g \in F_N$ we have $g(X, Y) := (gX, gY)$. The space $\partial^2 F_N$ comes equipped with the canonical “flip” map $\sigma : \partial^2 F_N \rightarrow \partial^2 F_N$, where $\sigma : (X, Y) \mapsto (Y, X)$ for $(X, Y) \in \partial^2 F_N$.

Definition 3.1 (Algebraic lamination). Let $N \geq 2$. An *algebraic lamination* on F_N is a closed F_N -invariant and flip-invariant subset $L \subseteq \partial^2 F_N$. Unless stated otherwise, we also require L to be nonempty.

For $X, Y \in \partial F_N$ such that $(X, Y) \in L$ we say that (X, Y) is a *leaf* of L . For $X \in \partial F_N$ we say that X is an *end* of L if there exists $Y \in \partial F_N, Y \neq X$ such that $(X, Y) \in L$.

Note that for an algebraic lamination L , the set of all ends of L is necessarily an F_N -invariant subset of ∂F_N .

Definition 3.2 (Laminary language). Let $\alpha : F_N \rightarrow \pi_1(\Gamma)$ be a marking on F_N and let $L \subseteq \partial^2 F_N$ be an algebraic lamination on F_N .

The *laminary language* \mathcal{L}_α of L with respect to α consists of all finite edge-paths γ in Γ such that some (equivalently, every) lift $\tilde{\gamma}$ of γ to $\tilde{\Gamma}$ is contained as a subpath in a $\tilde{\Gamma}$ -geodesic from X to Y for some $(X, Y) \in L$.

Thus we can regard \mathcal{L}_α as a set of words in the finite alphabet $E\Gamma$. For the case where α is a marking corresponding to some free basis A of F_N (so that Γ is a wedge of N circles corresponding to elements of A), we in fact identify $E\Gamma$ with $A \cup A^{-1}$ and think of \mathcal{L}_α as a set of words over the alphabet $A \cup A^{-1}$.

It turns out (see [16]) that an algebraic lamination is uniquely determined by its laminary language and that one can explicitly characterize which languages occur as laminary languages of algebraic laminations:

Proposition 3.3. [16] *Let $N \geq 2$. Let $\alpha : F_N \rightarrow \pi_1(\Gamma)$ be a marking on F_N .*

- (1) *Let $L, L' \subseteq \partial^2 F_N$ be algebraic laminations on F_N . Then $L = L'$ if and only if $\mathcal{L}_\alpha = \mathcal{L}'_\alpha$.*
- (2) *Let \mathcal{L} be a set of finite reduced edge-paths in Γ . Then $\mathcal{L} = \mathcal{L}_\alpha$ for some algebraic lamination L on F_N if and only if \mathcal{L} is closed under inversion, under taking subpaths and is bi-extendable, that is, for every nontrivial path $\gamma \in \mathcal{L}$ there exists a path $\gamma' = \gamma_1 \gamma \gamma_2 \in \mathcal{L}$ such that γ_1 and γ_2 are nontrivial.*

If γ is an edge-path (possibly infinite) in a graph Γ , we denote by $Sub(\gamma)$ the set of all finite subpaths of γ . If Ω is a set of edge-paths in Γ we put

$$Sub(\Omega) := \cup_{\gamma \in \Omega} Sub(\gamma).$$

We also need the operation of “diagonal extension” for subsets of $\partial^2 F_N$.

Definition 3.4 (Diagonal extension). For $R \subseteq \partial^2 F_N$ the *diagonal extension* of R , denoted $diag(R)$, is defined as

$$diag(R) = \{(X, Y) \in \partial^2 F_N : \text{there exists } m \geq 1, Z_0 = X, Z_1, \dots, Z_m = Y, \\ \text{such that } (Z_i, Z_{i+1}) \in R \text{ for } 0 \leq i \leq m-1\}.$$

Using $m = 1$ in the above definition we see that $R \subseteq diag(R)$ for every $R \subseteq \partial^2 F_N$. Note that, a priori, if $L \subseteq \partial^2 F_N$ is an algebraic lamination, then $diag(L)$ need not be an algebraic lamination, since $diag(L)$ may fail to be a closed subset of $\partial^2 F_N$.

In [17], Coulbois, Hilion and Lustig define, given an arbitrary $T \in \overline{cv}_N$, the “dual” or “zero” lamination of T :

Definition 3.5 (Dual lamination). Let $T \in \overline{cv}_N$ and let $\alpha : F_N \rightarrow \pi_1(\Gamma)$ be a marking on F_N . The *zero lamination* or *dual lamination* $L(T) \subseteq \partial^2 F_N$ of T consists of all $(X, Y) \in \partial^2 F_N$ with the following property:

For every finite subpath $[p, q]$ of the $\tilde{\Gamma}$ -geodesic from X to Y and for every $\varepsilon > 0$ there exists a cyclically reduced path w in Γ such that the projection γ of $[p, q]$ to Γ is a subpath of w and such that $\|w\|_T \leq \varepsilon$.

It is shown in [17] that $L(T) \subseteq \partial^2 F_N$ is indeed an algebraic lamination on F_N and that the above definition does not depend on the choice of a marking α . It is also clear that $L(T)$ depends only on the projective class $[T]$ of T in \overline{cv}_N .

Note that $L(T)$ can be described by specifying its laminary language with respect to a marking α . Namely, $L(T)$ is the unique algebraic lamination on F_N whose laminary language with respect to α is

$$\bigcap_{\varepsilon > 0} \cup \{Sub(w) : \|w\|_T \leq \varepsilon, \text{ and } w \text{ is a cyclically reduced path in } \Gamma\}.$$

There is an alternative description of $L(T)$ (in terms of the so-called map Q), obtained in [17], and stated in Proposition 4.1 below, which implies the following:

Proposition 3.6. *Let $N \geq 2$ and $T \in \overline{cv}_N$ be a tree with dense F_N -orbits. Then $L(T) = diag(L(T))$.*

Algebraic laminations on F_N and their generalizations also naturally arise in other contexts related to $Out(F_N)$, such as the proof of the Tits Alternative for $Out(F_N)$ by Bestvina, Feighn and Handel [7, 8], and in the study of geodesic currents on free groups [32].

3.2. Bestvina-Feighn-Handel laminations. For the remainder of this section we assume that $N \geq 3$ and that $\varphi \in \text{Out}(F_N)$ is a hyperbolic iwip. In this situation there are several natural algebraic laminations on F_N associated to φ .

The first one was introduced by Bestvina, Feighn and Handel in [6].

Definition 3.7. Let $f : \Gamma \rightarrow \Gamma$ be a train-track representative of φ and let $\alpha : F_N \rightarrow \pi_1(\Gamma)$ be the corresponding marking. Recall that the marking α defines an F_N -equivariant identification between ∂F_N and $\partial \tilde{\Gamma}$ and that we use this identification to write $\partial F_N = \partial \tilde{\Gamma}$.

The *Bestvina-Feighn-Handel lamination* $L_{BFH}(\varphi) \subseteq \partial^2 F_N$ of φ consists of all $(X, Y) \in \partial^2 F_N$ with the following property:

For every finite subpath $[p, q]$ of the geodesic in $\tilde{\Gamma}$ from X to Y there exists $n \geq 1$ and $e \in E\Gamma$ such that the projection γ of $[p, q]$ to Γ is a subpath of $f^n(e)$.

In terminology of [6], the lamination $L_{BFH}(\varphi)$ (respectively $L_{BFH}(\varphi^{-1})$) is called the *stable* (respectively *unstable*) lamination of φ .

One can show that the above definition does not depend on the choice of a train-track representative of φ and, moreover, that $L_{BFH}(\varphi) \subseteq \partial^2 F_N$ is an algebraic lamination on F_N . Moreover, it is not hard to see, in view of irreducibility properties of φ and of f , that in the above definition one can choose the edge $e_0 = e \in E\Gamma$ independent of X, Y and of $[p, q]$, provided we allow γ to be a subpath of $f^n(e_0)$ or of $f^n(e_0^{-1})$. Thus $L_{BFH}(\varphi)$ is the (unique) algebraic lamination with the laminary language

$$L_{BFH}(\varphi) = \cup_{n \geq 1} \cup_{e \in E\Gamma} \text{Sub}(f^n(e)) = \cup_{n \geq 1} (\text{Sub}(f^n(e_0)) \cup \text{Sub}(f^n(e_0^{-1})))$$

where $e_0 \in E\Gamma$ is any chosen in advance edge of Γ .

3.3. Attracting and repelling trees associated to iwips. The following fact is well-known [6, 19, 39]:

Proposition-Definition 3.8. Let $N \geq 2$ and let $\varphi \in \text{Out}(F_N)$ be an iwip. Then:

- (1) The action of φ on \overline{CV}_N has exactly two distinct fixed points. One of these fixed points, denoted $[T_+] = [T_+(\varphi)]$, has the property that $T_+\varphi = \lambda_+ T_+$ for some $\lambda_+ = \lambda_+(\varphi) > 1$ and the other fixed point, denoted $[T_-] = [T_-(\varphi)]$ has the property that $T_-\varphi = \frac{1}{\lambda_-} T_-$ for some $\lambda_- = \lambda_-(\varphi) > 1$.
- (2) The number λ_+ is equal to the Perron-Frobenius eigenvalue of any train-track representative of φ .
- (3) We have $[T_-(\varphi)] = [T_+(\varphi^{-1})]$ and $\lambda_-(\varphi) = \lambda_+(\varphi^{-1})$.
- (4) If $N \geq 3$ and φ is hyperbolic, then the action of F_N on $T_+(\varphi)$ is free and has dense orbits.

The point $[T_+]$ (or sometimes any representative $T_+ \in \overline{CV}_N$ of $[T_+]$) is called the *attracting tree* of φ or the *forward limit tree* of φ . Similarly, point $[T_-]$ (or sometimes any representative $T_- \in \overline{CV}_N$ of $[T_-]$) is called the *repelling tree* of φ or the *backward limit tree* of φ .

Part (4) of the above statement, together with Proposition 3.6 immediately imply:

Corollary 3.9. Let $N \geq 3$ and let $\varphi \in \text{Out}(F_N)$ be a hyperbolic iwip. Then $L(T_+(\varphi)) = \text{diag}(L(T_+(\varphi)))$.

The tree $T_+(\varphi)$ can be more explicitly understood using a train-track representative of φ (see [19]), but we will not need such a description in this paper.

Remark 3.10. If $\Phi \in \text{Aut}(F_N)$ is any automorphism, then Φ is a quasi-isometry of F_N and hence Φ induces a homeomorphism of ∂F_N and therefore a homeomorphism $\Phi : \partial^2 F_N \rightarrow \partial^2 F_N$ of $\partial^2 F_N$ as well. Moreover an inner automorphism acts on $\partial^2 F_N$ as a translation by an element of F_N . Thus if $\Phi_1, \Phi_2 \in \text{Out}(F_N)$ have the same outer automorphism class $\varphi \in \text{Out}(F_N)$, then the actions of Φ_1 and Φ_2 differ by the translation by an element of F_N .

If $\varphi \in \text{Out}(F_N)$ is a hyperbolic iwip and if $\Phi \in \text{Aut}(F_N)$ is a representative of φ , then the laminations $L_{BFH}(\varphi)$ and $L(T_\pm(\varphi))$ are Φ -invariant, that is $L_{BFH}(\varphi) = \Phi(L_{BFH}(\varphi)) = \Phi^{-1}(L_{BFH}(\varphi))$ and $\Phi(L(T_\pm(\varphi))) = \Phi^{-1}(L(T_\pm(\varphi))) = L(T_\pm(\varphi))$. Since, as algebraic laminations, $L_{BFH}(\varphi)$ and $L(T_\pm(\varphi))$ are also F_N -invariant, in view of the above facts, we will sometimes also say that $L_{BFH}(\varphi)$ and $L(T_\pm(\varphi))$ are φ -invariant (without invoking a specific lift of φ to $\text{Aut}(F_N)$).

In [35] we established a precise relationship between $L_{BFH}(\varphi)$ and $L(T_-)$:

Proposition 3.11. *Let $N \geq 3$ and let $\varphi \in \text{Out}(F_N)$ be an hyperbolic iwip. Then*

$$L(T_-(\varphi)) = \text{diag}(L_{BFH}(\varphi)).$$

The fact that $L_{BFH}(\varphi) \subseteq L(T_-(\varphi))$ is fairly straightforward. Hence, since $L(T_-(\varphi)) = \text{diag}(L(T_-(\varphi)))$ by Proposition 3.6, it follows that $\text{diag}(L_{BFH}(\varphi)) \subseteq L(T_-(\varphi))$. Showing that this last inclusion is actually an equality requires considerable extra work.

It is well-known (e.g. it easily follows from the results of [35]) that for a hyperbolic iwip φ the laminations $L(T_+)$ and $L(T_-)$ are disjoint in the following strong sense:

Proposition 3.12. *Let $N \geq 3$ and let $\varphi \in \text{Out}(F_N)$ be a hyperbolic iwip. Then $L(T_+) \cap L(T_-) = \emptyset$. Moreover, if $(X, Y) \in L(T_+)$ then there does not exist $Z \in \partial F_N$ such that $(X, Z) \in L(T_-)$; that is, the laminations $L(T_+)$ and $L(T_-)$ have no common ends.*

4. THE INDEX THEORY FOR TREES AND AUTOMORPHISMS

In this section we will discuss several notions of index for trees and automorphisms and the connections between these notions. We refer the reader to [15] for a more comprehensive review of this material.

4.1. The \mathcal{Q} -map and \mathcal{Q} -index. Let $T \in \overline{\text{cv}}_N$ be a tree with dense F_N -orbits (e.g. $T = T_+(\varphi)$, where $\varphi \in \text{Out}(F_N)$ is an hyperbolic iwip). We denote by \overline{T} the metric completion of T and put $\widehat{T} = \overline{T} \cup \partial T$, where ∂T is the hyperbolic boundary of T . Note that \widehat{T} comes equipped with a canonical F_N -action by homeomorphisms, and that the restriction of this action to \overline{T} gives an action of F_N on \overline{T} by isometries. Levitt and Lustig [39] introduced an F_N -equivariant map $\mathcal{Q} : \partial F_N \rightarrow \widehat{T}$ that turned out to be closely related to the dual lamination $L(T)$. The precise definition of the map \mathcal{Q} is not relevant for this paper, but we do need the following fact established in [17]:

Proposition 4.1. *Let $N \geq 2$ and let $T \in \overline{\text{cv}}_N$ be a tree with dense F_N -orbits. Then for $X, Y \in \partial F_N$, $X \neq Y$ we have $(X, Y) \in L(T)$ if and only if $\mathcal{Q}(X) = \mathcal{Q}(Y)$. Moreover, if $X \in \partial F_N$ and $P := \mathcal{Q}(X) \in \partial T$ then $\mathcal{Q}^{-1}(P) = \{X\}$.*

Proposition 4.1 immediately implies Proposition 3.6 above saying that for any $T \in \overline{cv}_N$ with dense F_N -orbits we have $L(T) = \text{diag}(L(T))$. Coulbois and Hilion [14] introduced a useful notion of a \mathcal{Q} -index in the above set-up. Again, let $T \in \overline{cv}_N$ be a tree with dense orbits. Let $P \in \hat{T}$. It is known in this case that $\text{Stab}_{F_N}(P)$ is a finitely generated (possibly trivial) free subgroup of F_N , so one has a natural inclusion $\partial \text{Stab}_{F_N}(P) \subseteq \partial F_N$. Define $\mathcal{Q}_r^{-1}(P) := \mathcal{Q}^{-1}(P) \setminus \partial \text{Stab}_{F_N}(P)$. It is easy to see that $\mathcal{Q}_r^{-1}(P)$ is a $\text{Stab}_{F_N}(P)$ -invariant subset of ∂F_N . Denote by $[P]$ the equivalence class of P in the quotient set $\mathcal{Q}_r^{-1}(P)/\text{Stab}_{F_N}(P)$. For $P \in \hat{T}$ define

$$(\clubsuit) \quad \text{ind}_{\mathcal{Q}}(P) := \#(\mathcal{Q}_r^{-1}(P)/\text{Stab}_{F_N}(P)) + 2 \text{rank}(\text{Stab}_{F_N}(P)) - 2.$$

It is easy to see that for any $P' \in [P]$ we have $\text{ind}_{\mathcal{Q}}(P') = \text{ind}_{\mathcal{Q}}(P)$. Thus we define $\text{ind}_{\mathcal{Q}}([P]) := \text{ind}_{\mathcal{Q}}(P)$. Finally, we can define the \mathcal{Q} -index of T :

Definition 4.2 (\mathcal{Q} -index of a tree). Let $T \in \overline{cv}_N$ be a tree with dense orbits. Define the \mathcal{Q} -index $\text{ind}_{\mathcal{Q}}(T)$ as

$$(*) \quad \text{ind}_{\mathcal{Q}}(T) := \sum_{[P] \in \hat{T}/F_N} \max\{0, \text{ind}_{\mathcal{Q}}([P])\}.$$

We will discuss in greater detail the case where $T \in \overline{cv}_N$ has dense orbits and corresponds to a free action. This is the only situation that is relevant for the proof of our main result, and the definition of $\text{ind}_{\mathcal{Q}}(T)$ becomes easier in this case.

Note that \overline{T} is a complete \mathbb{R} -tree containing T as a subtree. For a point $P \in \overline{T}$ the *valence* $\text{val}_{\overline{T}}(P)$ of P in \overline{T} is the number of connected components of $\overline{T} \setminus P$, called *directions* at P . A direction at P can be also identified with an equivalence class of non-degenerate segments $[P, P']$ where $P \in \overline{T}$, $P \neq P'$. Here two such segments $[P, P']$, $[P, P'']$ are equivalent if $[P, P'] \cap [P, P''] = [P, S]$ for some $S \neq P$.

We say that $P \in \overline{T}$ is a *branch point* if $\text{val}_{\overline{T}}(P) \geq 3$. Note that completion points always have degree 1, so that if $P \in \overline{T}$ is a branch point then $P \in T$.

We need the following basic lemma:

Lemma 4.3. *Let $T \in cv_N$ be a free F_N -tree. Then the action of F_N on \overline{T} is also free.*

Proof. Let $P \in \overline{T} \setminus T$ be a completion point. It is not hard to check that $\text{val}_{\overline{T}}(P) = 1$. Let $g \in F_N$ be such that $gP = P$. The stabilizer $\text{Stab}_{F_N}(P)$ acts on the set of directions at P . Since \overline{T} has exactly one direction at P , it follows that there exists $x \in T$ such that the segment $[P, x]$ is fixed pointwise by g . In particular, $gx = x$. Since F_N acts freely on T by assumption, it follows that $g = 1$. Thus F_N acts freely on \overline{T} , as claimed. \square

Corollary 4.4. *Let $T \in cv_N$ be a free F_N -tree with dense F_N -orbits. Then for any $P \in \overline{T}$ we have*

$$\text{ind}_{\mathcal{Q}}(P) = \#(\mathcal{Q}^{-1}(P)) - 2,$$

and

$$(**) \quad \text{ind}_{\mathcal{Q}}(T) = \sum_{[P] \in \overline{T}/F_N} \max\{0, \#(\mathcal{Q}^{-1}(P)) - 2\}$$

Proof. Proposition 4.1 implies that $\max\{0, \text{ind}_{\mathcal{Q}}(P)\} = 0$ for every $P \in \partial T$. Therefore in the definition of $\text{ind}_{\mathcal{Q}}(T)$ we only need to sum up over $[P] \in \overline{T}/F_N$.

By Lemma 4.3 we have $Stab_{F_N}(P) = \{1\}$ for every $P \in \overline{T}$, and hence we can ignore the stabilizers in the formula (\clubsuit) above defining $\text{ind}_{\mathcal{Q}}(P)$. Thus in this case for every $P \in \overline{T}$

$$\text{ind}_{\mathcal{Q}}(P) = \#(\mathcal{Q}^{-1}(P)) - 2.$$

In particular, $\max\{0, \text{ind}_{\mathcal{Q}}(P)\} = 0$ provided $\#(\mathcal{Q}^{-1}(P)) \leq 2$. Hence

$$\text{ind}_{\mathcal{Q}}(T) = \sum_{[P] \in \overline{T}/F_N} \max\{0, \#(\mathcal{Q}^{-1}(P)) - 2\}$$

and the only $[P] \in \overline{T}/F_N$ that contribute positive terms to the above sum are those points such that $\#(\mathcal{Q}^{-1}(P)) \geq 3$. Formula (**) now follows immediately from the definition of $\text{ind}_{\mathcal{Q}}(T)$. \square

The following important general fact was recently established by Coulbois and Hilion in [14]

Proposition 4.5. *Let $N \geq 2$ and let $T \in \overline{cv}_N$ be a tree with dense F_N -orbits. Then $\text{ind}_{\mathcal{Q}}(T) \leq 2N - 2$.*

For the applications in this paper we will only need a special case of Proposition 4.5, for the situation where $T = T_{\pm}(\varphi)$, with $\varphi \in \text{Out}(F_N)$ being an hyperbolic iwip. For that case the conclusion of Proposition 4.5, given by Proposition 4.11 below, is substantially easier to obtain.

4.2. Index of an automorphism. We need to use the notion of an *index* of an automorphism introduced by Gaboriau, Jaeger, Levitt and Lustig in [20]. For simplicity (since in the applications we only need to consider this case), we will define the index only for atoroidal automorphisms. Note that if $\varphi \in \text{Out}(F_N)$ is atoroidal, then for every representative $\Phi \in \text{Aut}(F_N)$ of φ the fixed subgroup $\text{Fix}(\Phi) \leq F_N$ of Φ is trivial (which is what makes defining the index of φ easier in this case).

It is proved in [20] that if $\Phi \in \text{Aut}(F_N)$ is arbitrary, then for every point $X \in \partial F_N$ fixed by Φ exactly one of the following occurs:

- The point X belongs to the boundary of the fixed subgroup $\text{Fix}(\Phi)$ of Φ .
- The point X is a local attractor for the action of Φ on $F_N \cup \partial F_N$.
- The point X is a local attractor for the action of Φ^{-1} on $F_N \cup \partial F_N$.

Denote by $a(\Phi)$ the number of local Φ -attractors in ∂F_N . We will also denote the set of local Φ -attractors in ∂F_N by $\text{Att}(\Phi)$.

Definition 4.6 (Index of an automorphism). Let $\Phi \in \text{Aut}(F_N)$ be an atoroidal automorphism. The *index* of Φ is

$$\text{ind}(\Phi) := \frac{a(\Phi)}{2} - 1.$$

Definition 4.7 (Index of an outer automorphism). For $x \in F_N$ denote by i_x the inner automorphism of F_N given by $i_x(h) = xhx^{-1}$, $h \in F_N$. Two automorphisms $\Phi, \Phi' \in \text{Aut}(F_N)$ are *isogredient* if there exists $x \in F_N$ such that

$$\Phi' = i_x \circ \Phi \circ i_{x^{-1}} = i_{x\Phi(x^{-1})} \circ \Phi.$$

If Φ, Φ' are isogredient, then their actions on ∂F_N are conjugate by a translation by an element of F_N , and hence $\text{ind}(\Phi) = \text{ind}(\Phi')$. However, in general if two automorphisms represent the same class in $\text{Out}(F_N)$, their indices may be different.

We denote the isogredience class of Φ by $[[\Phi]]$. Note that the outer automorphism class Φ partitions as a disjoint union of isogredience classes. An isogredience class $[[\Phi]]$ is *essential* if $\text{ind}(\Phi) > 0$ for some (equivalently, any) $\Phi \in [[\Phi]]$. We denote by \mathcal{E}_φ the set of all essential isogredience classes of representatives of φ in $\text{Aut}(F_N)$.

For an atoroidal $\varphi \in \text{Out}(F_N)$ define the *index* of φ as

$$\text{ind}(\varphi) = \sum_{[[\Phi]] \in \mathcal{E}_\varphi} \text{ind}(\Phi).$$

We need the following notion which, for simplicity, we again will only define for atoroidal elements of $\text{Out}(F_N)$, since this is the only case needed for our applications.

Definition 4.8 (Forward rotationless automorphisms). Let $\varphi \in \text{Out}(F_N)$ be an atoroidal element. Following [15], we say that φ is *forward rotationless* (FR) if the following hold:

- (1) For any representative $\Phi \in \text{Aut}(F_N)$ every Φ -periodic point in ∂F_N is fixed by Φ .
- (2) If $\Psi \in \text{Aut}(F_N)$ is a representative of φ^n for some $n > 0$ and $\text{ind}(\Psi) > 0$ then there exists a representative $\Phi \in \text{Aut}(F_N)$ of φ such that $\Psi = \Phi^n$.

Proposition 3.3 in [15] shows that any element of $\text{Out}(F_N)$ admits a positive power which is FR; in particular this fact holds for atoroidal elements of $\text{Out}(F_N)$.

4.3. Geometric index, eigenrays and homotheties. In [21] Gaboriau and Levitt introduced the notion of a “geometric index” for any $T \in \overline{\text{cv}}_N$. In the same paper they proved that for any free F_N -tree $T \in \overline{\text{cv}}_N$ the number of F_N -orbits of branch-points of T is finite and is bounded above by $2N - 2$. As before, for a point $P \in T$ we will denote by $[P]$ the F_N -orbit of P .

For simplicity, we will only define geometric index for free F_N -trees, since that is the only case relevant for our arguments.

Definition 4.9 (Geometric index). Let $T \in \overline{\text{cv}}_N$ be a free F_N -tree. Define the *geometric index* $\text{ind}_{\text{geom}}(T)$ as

$$\text{ind}_{\text{geom}}(T) := \sum_{[P]: \text{val}(P) \geq 3} [\text{val}_T(P) - 2].$$

In particular, Gaboriau and Levitt proved in [21] that for any $T \in \overline{\text{cv}}_N$ we have $\text{ind}_{\text{geom}}(T) \leq 2N - 2$, and we will use this fact below for the trees T_\pm of atoroidal iwips.

We now briefly review the notions of homotheties and eigenrays associated with atoroidal iwips. We refer the reader to [20, 40, 34] for the detailed background information on this topic.

Suppose that $\varphi \in \text{Out}(F_N)$ is an atoroidal iwip and $T_\pm \in \overline{\text{cv}}_N$ are its attracting/repelling trees. Thus in $\overline{\text{cv}}_N$ we have $T_+\varphi = \lambda_+T_+$ and $T_-\varphi^{-1} = \lambda_-T_-$ with $\lambda_+, \lambda_- > 1$. Let $\Phi \in \text{Aut}(F_N)$ be a representative of φ . Since $T_+\Phi = \lambda_+T_+$ in $\overline{\text{cv}}_N$, this means that the F_N -trees $T_+\Phi$ and λ_+T_+ are F_N -equivariantly isometric. Taking into account the definition of the action of $\text{Aut}(F_N)$ on $\overline{\text{cv}}_N$, this fact translates into the existence of a *homothety* $H = H_\Phi : T_+ \rightarrow T_+$ which multiplies all distances by λ_+ and such that $H(xp) = \Phi(x)H(p)$ for all $p \in T_+$ and $x \in F_N$. In this case we say that the homothety H *represents* Φ , and, also, that H *represents* φ .

For every left Φ of φ to $\text{Aut}(F_N)$ the homothety $H = H_\Phi$ has a unique fixed point $C_H \in \overline{T_+}$ called the *center* of H . (Note that sometimes C_H is a completion point). In general H permutes the set of directions at C_H in $\overline{T_+}$. Since every point of $\overline{T_+}$ has finite valence, there is a positive power H^t of H (which represents Φ^t) such that H^t fixes all directions at C_H in $\overline{T_+}$. Then for any direction \mathbf{d} at C_H there exists a unique geodesic ray ρ in $\overline{T_+}$ emanating from C_H in direction \mathbf{d} and such that $H^t \rho = \rho$. The ray ρ is called an *eigenray* of H and is sometimes denoted $\rho_{\mathbf{d}}$.

Moreover, it is known that any two homotheties of T_+ representing φ differ by a translation by an element of F_N . Since F_N acts freely on $\overline{T_+}$, this implies that if two homotheties H_1, H_2 of T_+ representing φ have the same center then $H_1 = H_2$. In a similar way, we can define the notions of homotheties of T_- representing lifts of φ^{-1} to $\text{Aut}(F_N)$ and of eigenrays for such homotheties.

The following proposition elucidates the connections between $\text{ind}_{\text{geom}}(T_+)$, $\text{ind}_{\mathcal{Q}}(T_-)$ and $\text{ind}(\varphi)$ (listed in Proposition 4.11 below). Proposition 4.10 is obtained in [35] and, a weaker form of this proposition follows from the proofs of Lemma 4.3 and Proposition 4.4 in [15].

Proposition 4.10. *Let $N \geq 3$ and let $\varphi \in \text{Out}(F_N)$ be an atoroidal iwip. Let $\mathcal{Q}_+ : \partial F_N \rightarrow \widehat{T_+}$ and $\mathcal{Q}_- : \partial F_N \rightarrow \widehat{T_-}$ be the \mathcal{Q} -maps. Let $\Phi \in \text{Aut}(F_N)$ be a representative of φ and let $H = H_\Phi$ be the corresponding homothety of T_+ . Let $\text{Br}(T_+)$ be the set of all branch-points in T_+ and let $Y(T_-)$ be the set of all $P' \in \overline{T_-}$ with $\#((\mathcal{Q}_-^{-1})(P')) \geq 3$.*

Then there exists an F_N -equivariant bijection $\Upsilon : \text{Br}(T_+) \rightarrow Y(T_-)$ such that the following hold:

- (1) *For every branch-point P of T_+ and the point $P' = \Upsilon(P) \in Y(T_-)$ we have:*
 - (a) *there is a (unique) homothety H' of T_+ representing φ with $C_{H'} = P$*
 - (b) *we have $\text{val}_{T_+}(P) = \#((\mathcal{Q}_-^{-1})(P')) = k \geq 3$*
 - (c) *there exists a bijection $\xi_P : X_i \mapsto \rho_i$ between the fiber $(\mathcal{Q}_-^{-1})(P') = \{X_1, \dots, X_k\}$ and the set of eigenrays $\{\rho_1, \dots, \rho_k\}$ of H' in T_+ emanating from P such that for $i = 1, \dots, k$ the point $\mathcal{Q}_+(X_i) \in \partial T_+$ is the end at infinity of the ray ρ_i .*
 - (d) *For any $w \in F_N$ and $t \geq 1$ the homothety wH^t fixes P if and only if every X_i , $i = 1, \dots, k$ is an attracting fixed point in ∂F_N of $i_w \circ \Phi^t$.*
- (2) *Conversely, if $P' \in \overline{T_-}$ is such that the fiber $(\mathcal{Q}_-^{-1})(P') = \{X_1, \dots, X_k\}$ has cardinality $k \geq 3$ then there exists a unique branch-point $P \in T_+$ of valence $k \geq 3$ such that \mathcal{Q}_+ maps X_1, \dots, X_k to the ends at infinity in ∂T_+ of the eigenrays ρ_1, \dots, ρ_k emanating from P in T_+ ; this point P is exactly $\Upsilon^{-1}(P')$.*

We can now connect all the three notions of index together (again see [15] for a detailed exposition regarding these connections). The following proposition is not new and is contained, at least implicitly, in several prior papers (e.g. it follows directly from the results of [15]), but we still think it is useful to provide an indication of the proof here.

Proposition 4.11. *Let $N \geq 3$ and let $\varphi \in \text{Out}(F_N)$ be an atoroidal iwip.*

Then the following hold:

- (1) *We have $\text{ind}_{\mathcal{Q}}(T_-) = \text{ind}_{\text{geom}}(T_+)$.*
- (2) *We have $\text{ind}_{\mathcal{Q}}(T_-) \leq 2N - 2$.*

(3) If we further assume that φ^{-1} is FR then

$$2\text{ind}(\varphi) = \text{ind}_{\mathcal{Q}}(T_-).$$

Sketch of proof. Part (1) of the proposition can be derived, by unpacking the definitions, from Proposition 4.10 above. Alternatively, part (1) is exactly Proposition 5.1 in [15]. The proof of Proposition 5.1 in [15] relies in part on the results of [14], but does not require the hard general results of [14] that are used in the proof of Proposition 4.5.

In [21] Gaboriau and Levitt proved that for any $T \in \overline{\text{cv}}_N$ we have $\text{ind}_{\text{geom}}(T) \leq 2N - 2$. This fact, together with part (1), implies part (2). Part (3) can again be derived from Proposition 4.10. An alternative proof of part (3), in a slightly more general setting, is given in Section 4 of [20] (see also Proposition 4.4 in [15] for yet another direct proof of (3)). \square

5. THE CANNON-THURSTON MAP

Definition 5.1 (Cannon-Thurston map). Let G be a word-hyperbolic group and let $H \leq G$ be a word-hyperbolic subgroup of G . Recall that $\overline{H} = H \cup \partial H$ and $\overline{G} = G \cup \partial G$ denote the hyperbolic compactifications of these groups. Following Mitra, we denote the inclusion of H into G as $\iota : H \rightarrow G$ (although usually we will suppress explicit mention of ι and will think of H as a subset of G).

Suppose there exists a map $\hat{\iota} : \partial H \rightarrow \partial G$ such that the function $J : \overline{H} \rightarrow \overline{G}$ defined by the formula

$$J : h \mapsto \iota(h), \text{ for } h \in H, \quad J : X \mapsto \hat{\iota}(X), \text{ for } X \in \partial H$$

is continuous.

Then $\hat{\iota}$ is called a *Cannon-Thurston map*.

Thus, by definition, if the Cannon-Thurston map exists, it is continuous. Definition 5.1 also means that if $\hat{\iota}$ exists, and $h_n \in H, X \in \partial H$ are such that $\lim_{n \rightarrow \infty} h_n = X$ in the topology of \overline{H} then $\lim_{n \rightarrow \infty} h_n = \hat{\iota}(X)$ in the topology of \overline{G} . Hence if the Cannon-Thurston map exists, it is unique. Also, Definition 5.1 implies that the Cannon-Thurston map $\hat{\iota} : \partial H \rightarrow \partial G$ (provided it exists) is H -equivariant. That is, for any $X \in \partial H$ and $h \in H$ we have $\hat{\iota}(hX) = h\hat{\iota}(X)$.

Remark 5.2. Suppose the Cannon-Thurston map $\hat{\iota} : \partial H \rightarrow \partial G$ exists. Then the image $\hat{\iota}(\partial H)$ is equal to the *limit set* $\Lambda(H)$ of H in ∂G . Here $\Lambda(H)$ is the set of all limits in ∂G of sequences of elements from H . The inclusion $\hat{\iota}(\partial H) \subseteq \Lambda(H)$ is obvious. For the opposite inclusion, let $S \in \Lambda(H)$. Then there exists a sequence $h_n \in H$ such that $\lim_{n \rightarrow \infty} h_n = S$ in the topology of $G \cup \partial G$. Since $H \cup \partial H$ is compact, after passing to a subsequence we may assume that h_n converges to some $X \in \partial H$ in the topology of $H \cup \partial H$. Therefore, by definition of the Cannon-Thurston map, we have $\hat{\iota}(X) = S$. Thus indeed, $\hat{\iota}(\partial H) = \Lambda(H)$.

As noted in the Introduction, in [47], Mitra proves that, given a short exact sequence

$$(\dagger) \quad 1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

of word-hyperbolic groups H, G, Q there does exist a Cannon-Thurston map $\hat{\iota} : \partial H \rightarrow \partial G$.

Remark 5.3. Suppose $H \leq G$ are hyperbolic groups, such that H is infinite, normal in G and the quotient group $Q = G/H$ is infinite, and such that $\widehat{\iota} : \partial H \rightarrow \partial G$ exists (e.g. if Q is also word-hyperbolic). Then the Cannon-Thurston map $\widehat{\iota} : \partial H \rightarrow \partial G$ is surjective. Indeed, $\widehat{\iota}(\partial H) = \Lambda(H)$, by Remark 5.2. Since H is infinite and normal in G , by a result of [36], we have $\Lambda(H) = \partial G$. Hence $\widehat{\iota}(\partial H) = \partial G$, as claimed.

We will need the following simple but useful lemma:

Lemma 5.4. *Let H, G, Q be word-hyperbolic groups as in (\dagger) . Let $g \in G$, so that the conjugation by g on H gives us an automorphism Φ of H : $\Phi(h) = ghg^{-1}, h \in H$. (Since Φ is a quasi-isometry of H , Φ induces a homeomorphism of ∂H , which we still denote by Φ .) Let $X \in \partial H$ and let $S = \widehat{\iota}(X) \in \partial G$. Then $gS = \widehat{\iota}(\Phi(X))$; that is $g\widehat{\iota}(X) = \widehat{\iota}(\Phi(X))$.*

Proof. Choose a sequence $h_n \in H$ such that $\lim_{n \rightarrow \infty} h_n = X$ in the topology of $H \cup \partial H$. By definition of $\widehat{\iota}$ it follows that $\lim_{n \rightarrow \infty} h_n = S$ in the topology of $G \cup \partial G$. In G we have $gh_n g^{-1} = \Phi(h_n)$ so that

$$gS = \lim_{n \rightarrow \infty} gh_n = \lim_{n \rightarrow \infty} gh_n g^{-1} = \lim_{n \rightarrow \infty} \Phi(h_n)$$

in the topology of $G \cup \partial G$. By definition of $\widehat{\iota}$ the last limit above is exactly $\widehat{\iota}(\Phi(X))$. Thus $gS = \widehat{\iota}(\Phi(X))$, as claimed. \square

6. MITRA'S LAMINATION AND FIBERS OF THE CANNON-THURSTON MAP

In [46], for a short exact sequence (\dagger) of word-hyperbolic groups, Mitra characterizes the fibers of the Cannon-Thurston map $\widehat{\iota} : \partial H \rightarrow \partial G$ in terms of the so-called “ending laminations”. We recall briefly this general description here and will then give a more precise statement for mapping tori of atoroidal iwips.

Given every $z \in \partial Q$, Mitra defines an “ending lamination” $\Lambda_z \subseteq \partial^2 H$. To define Λ_z , Mitra starts with choosing a quasi-isometric section $r : Q \rightarrow G$ (he later proves that the specific choice of r does not matter). Then given any $z \in \partial Q$, take a geodesic ray in Q towards z and let z_n be the point at distance n from the origin on that ray. Lift z_n to G by the section r to get an element $g_n = r(z_n) \in G$. Conjugation by g_n gives an automorphism φ_n of H defined as $\varphi_n(h) = g_n h g_n^{-1}$, $h \in H$. Now pick any non-torsion element $h \in H$. Then look at all bi-infinite geodesics $\gamma = (X, Y)$ in the Cayley graph of H (where $X, Y \in \partial H$) such that there exist sequences x_n, y_n of vertices of with $x_n \rightarrow X$ and $y_n \rightarrow Y$ with the property that the segment $[x_n, y_n] \subseteq \gamma$ is labelled by a word in H which is a “subword” (in a properly quasified sense) of a “cyclically reduced” H -geodesic form of $\varphi_{k_n}(h)$ with $k_n \rightarrow \infty$ as $n \rightarrow \infty$. For a fixed h , the collection of all such $(X, Y) \in \partial^2 H$ is denoted $\Lambda_{z,h}$. Finally, put $\Lambda_z = \cup_h \Lambda_{z,h}$ where h varies over all non-torsion elements of H . The main result of [46] says that, for a short exact sequence (\dagger) of word-hyperbolic groups, if $X, Y \in \partial H$, $X \neq Y$ then $\widehat{\iota}(X) = \widehat{\iota}(Y)$ if and only if $(X, Y) \in \Lambda_z$ for some $z \in \partial Q$.

If Q is free (e.g. infinite cyclic), then one can choose the section $r : Q \rightarrow G$ to be an injective homomorphism, and think of $r : Q \rightarrow G$ as an inclusion.

By the Bestvina-Feighn Combination Theorem [4], if $\varphi \in \text{Out}(F_N)$ is a hyperbolic automorphism and $\Phi \in \text{Aut}(F_N)$ is a representative of φ then the mapping

torus group of Φ ,

$$M_\Phi = F_N \rtimes_\Phi \langle t \rangle = \langle F_N, t | tht^{-1} = \Phi(h), h \in F_N \rangle$$

is word-hyperbolic. In the case, for $H = F_N$, $Q = \langle t \rangle \cong \mathbb{Z}$ and $G = M_\Phi$, we get a short exact sequence of the above type (‡):

$$1 \rightarrow F_N \rightarrow M_\Phi \rightarrow \langle t \rangle \rightarrow 1.$$

Note also that replacing t by $t_1 = ut$, where $u \in F_N$ is arbitrary, rewrites the relation $tht^{-1} = \Phi(h)$ into $t_1ht = u\Phi(h)u^{-1}$. Thus M_Φ and the inclusion $F_N \leq M_\Phi$ depend only on the class outer automorphism φ of Φ . Since F_N , M_Φ and \mathbb{Z} are word-hyperbolic, Mitra's result [47] implies that the inclusion $F_N \leq M_\Phi$ does extend to a continuous Cannon-Thurston map $\hat{\iota}: \partial F_N \rightarrow \partial M_\Phi$.

For the infinite cyclic group $Q = \langle t \rangle$ its hyperbolic boundary consists of exactly two distinct points, t^∞ and $t^{-\infty}$. We can take $r(t^n) = t^n$. Then $\varphi_n = \Phi^n$. Translating Mitra's definition of Λ_z , for $z = t^\infty$, to this context yields the definition of $\Lambda_{t^\infty} = \Lambda_\varphi$ (where $\varphi \in \text{Out}(F_N)$ is the outer automorphism class of Φ) given in Definition 6.1 below. Similarly, for $z = t^{-\infty}$ we get $\Lambda_{t^{-\infty}} = \Lambda_{\varphi^{-1}}$.

We now make these statements and definitions more precise for the case of $G = M_\Phi$.

Definition 6.1 (Mitra's lamination of a hyperbolic automorphism). Let $N \geq 3$ and let $\varphi \in \text{Out}(F_N)$ be a hyperbolic automorphism.

Let $\alpha: F_N \rightarrow \pi_1(\Gamma)$ be a marking on F_N and let $f: \Gamma \rightarrow \Gamma$ be a topological representative of φ .

For every $h \in F_N$, $h \neq 1$ let w_h be a cyclically reduced closed path in Γ corresponding to the conjugacy class $[\alpha(h)]$. For $n \geq 1$ let $W_{h,n}$ be a cyclically reduced closed path in Γ representing $[f^n(w_h)]$.

Put $\Lambda_{\varphi,h} \subseteq \partial^2 F_N$ to be the set of all $(X, Y) \in \partial^2 F_N$ such that for every finite subpath $[p, q]$ of the $\tilde{\Gamma}$ -geodesic from X to Y the projection γ of $[p, q]$ to Γ is a subpath of a cyclic permutation of $W_{h,n}$ for some $n \geq 1$. Put $\Lambda_\varphi := \cup_{h \in F_N, h \neq 1} \Lambda_{\varphi,h}$.

Note that $\Lambda_{\varphi,h}$, after symmetrization (i.e. being made flip-invariant) is exactly what was termed the lamination generated by the set of conjugacy classes $\varphi^t(h)$, $t = 1, 2, \dots$, in [35].

It follows from the general results of [46] (and also from [35]) that the above definition of Λ_φ does not depend on the choice of α and f .

The general result of Mitra [46] applied to M_Φ implies:

Theorem 6.2. [46] *Let $\varphi \in \text{Out}(F_N)$ be a hyperbolic element and let $\Phi \in \text{Aut}(F_N)$ be a representative of φ . Let $\hat{\iota}: \partial F_N \rightarrow \partial M_\Phi$ be the Cannon-Thurston map.*

Then for $(X, Y) \in \partial^2 F_N$ we have $\hat{\iota}(X) = \hat{\iota}(Y)$ if and only if

$$(X, Y) \in \Lambda_\varphi \cup \Lambda_{\varphi^{-1}}.$$

Remark 6.3. It is not hard to see directly from the definitions that the various laminations described above are preserved by passing to a positive power of φ . Namely if $\varphi \in \text{Out}(F_N)$ is a hyperbolic iwip, and $k \geq 1$ then $L_{BFH}(\varphi) = L_{BFH}(\varphi^k)$, $T_\pm(\varphi) = T_\pm(\varphi^k)$, $L(T_\pm(\varphi)) = L(T_\pm(\varphi^k))$ and $\Lambda_\varphi = \Lambda_{\varphi^k}$.

Establishing the following fact is the first step in the proof of the main result of this paper:

Proposition 6.4. *Let $N \geq 3$ and $\varphi \in \text{Out}(F_N)$ be an atoroidal iwip. Then $\Lambda_\varphi = L(T_-(\varphi))$.*

Proof. Let $f : \Gamma \rightarrow \Gamma$ be a train-track representative of φ and let $\alpha : F_N \rightarrow \pi_1(\Gamma)$ be the corresponding marking. After possibly replacing φ by its power, we may assume that for any two edges $e, e' \in E\Gamma$ the path $f(e)$ passes through e' or $(e')^{-1}$.

Recall that $T_-\varphi = \frac{1}{\lambda_-}T_-$ and hence for every $h \in F_N$ we have

$$\|\varphi^n(h)\|_{T_-} = \frac{1}{\lambda_-^n} \|h\|_{T_-} \rightarrow 0$$

as $n \rightarrow \infty$. Therefore the definitions of $\Lambda_{\varphi,h}$ and of $L(T_-)$ imply that for every $h \in F_N, h \neq 1$ we have $\Lambda_{\varphi,h} \subseteq L(T_-)$. Hence, by definition of L_φ , it follows that $\Lambda_\varphi \subseteq L(T_-)$. A symmetric argument shows that $\Lambda_{\varphi^{-1}} \subseteq L(T_+)$.

Since $\Lambda_\varphi \subseteq L(T_-)$ and since, by Proposition 3.6, $\text{diag}(L(T_-)) = L(T_-)$, it follows that $\text{diag}(\Lambda_\varphi) \subseteq L(T_-)$ and, by symmetry, $\text{diag}(\Lambda_{\varphi^{-1}}) \subseteq L(T_+)$.

Now choose $h \in F_N, h \neq 1$ to be arbitrary and let w_h be a cyclically reduced closed edge-path in Γ representing the conjugacy class $[\alpha(h)]$. Let $W_{h,n}$ be the cyclically reduced form of $f^n(w_h)$. Let m be the simplicial length of w_h . Thus $W_{h,n}$ is a concatenation of $k_n \leq m$ paths $W_{h,n} = \gamma_{1,n} \dots, \gamma_{k_n,n}$, where each $\gamma_{i,j}$ is a subpath of $f^n(e)$ for some $e \in E\Gamma$. Since φ is hyperbolic and has no periodic conjugacy classes, we have $|W_{h,n}| \rightarrow \infty$ as $n \rightarrow \infty$. Since $k_n \leq m$ is bounded, after possibly replacing the paths $W_{h,n}$ by their cyclic permutations, we may assume that $|\gamma_{1,n}| \rightarrow \infty$ as $n \rightarrow \infty$ and that $\gamma_{1,n}$ is a subpath of $f^n(e_n)$ for some $e_n \in E\Gamma$. The assumptions on f now imply that for every edge $e \in E\Gamma$ and every $t \geq 1$ there is $n \geq 1$ such that $f^t(e)$ is a subpath of $f^n(e_n)$ or of $f^n(e_n^{-1})$ and hence is a subpath of $W_{h,n}$ or $W_{h,n}^{-1}$. It then follows from the definitions of $L_{BFH}(\varphi)$ and of $\Lambda_{\varphi,h}$ that $L_{BFH}(\varphi) \subseteq \Lambda_{\varphi,h}$. Hence $L_{BFH}(\varphi) \subseteq \Lambda_\varphi$ and therefore $\text{diag}(L_{BFH}(\varphi)) \subseteq \text{diag}(\Lambda_\varphi)$.

Thus

$$\text{diag}(L_{BFH}(\varphi)) \subseteq \text{diag}(\Lambda_\varphi) \subseteq L(T_-).$$

Since, by Proposition 3.11, $\text{diag}(L_{BFH}(\varphi)) = L(T_-)$, it follows that $\text{diag}(\Lambda_\varphi) = L(T_-)$. By symmetry, we also get $\text{diag}(\Lambda_{\varphi^{-1}}) = L(T_+)$. Proposition 3.12 now implies that

$$(\dagger) \quad \text{diag}(\Lambda_\varphi \cup \Lambda_{\varphi^{-1}}) = \text{diag}(\Lambda_\varphi) \sqcup \text{diag}(\Lambda_{\varphi^{-1}}).$$

Let R be the set of all $(X, Y) \in \partial^2 F_N$ such that $\hat{i}(X) = \hat{i}(Y)$. Clearly, $R = \text{diag}(R)$. We also know, by Theorem 6.2, that $R = \Lambda_\varphi \cup \Lambda_{\varphi^{-1}}$. Now (\dagger) implies that $\text{diag}(\Lambda_\varphi) = \Lambda_\varphi$ and $\text{diag}(\Lambda_{\varphi^{-1}}) = \Lambda_{\varphi^{-1}}$. Since we already know that $\text{diag}(\Lambda_\varphi) = L(T_-)$, we conclude that $\Lambda_\varphi = L(T_-)$, as claimed. \square

Proposition 6.4 and Theorem 6.2 immediately imply:

Corollary 6.5. *Let $\varphi \in \text{Out}(F_N)$ be an atoroidal iwip and let $\Phi \in \text{Aut}(F_N)$ be a representative of φ . Let $\hat{i} : \partial F_N \rightarrow \partial M_\Phi$ be the Cannon-Thurston map.*

Then for $(X, Y) \in \partial^2 F_N$ we have $\hat{i}(X) = \hat{i}(Y)$ if and only if

$$(X, Y) \in L(T_+(\varphi)) \cup L(T_-(\varphi)).$$

7. THE $2N$ -BOUND FOR THE SIZE OF THE FIBERS OF THE CANNON-THURSTON MAP

We can now prove Theorem A from the Introduction:

Theorem 7.1. *Let $N \geq 3$ and let $\varphi \in \text{Out}(F_N)$ be a hyperbolic iwip. Let $\Phi \in \text{Aut}(F_N)$ be a representative of φ and let $M_\Phi = F_N \rtimes_\Phi \mathbb{Z}$ be the mapping torus group of Φ . Let $\hat{\iota} : \partial F_N \rightarrow \partial M_\Phi$ be the Cannon-Thurston map.*

Then for every $S \in \partial M_\Phi$ we have $\deg(S) \leq 2N$.

Proof. We say that a point $X \in \partial F_N$ is φ -relevant (respectively, φ^{-1} -relevant) if there exists $Y \in \partial F_N$, $Y \neq X$ such that $(X, Y) \in \Lambda_\varphi$ (respectively $(X, Y) \in \Lambda_{\varphi^{-1}}$).

Let $S \in M_\Phi$ and let $X \in \partial F_N$ be such that $\hat{\iota}(X) = S$.

By Theorem 6.2 for $(X, Y) \in \partial^2 F_N$ we have $\hat{\iota}(X) = \hat{\iota}(Y)$ if and only if

$$(!) \quad (X, Y) \in \Lambda_\varphi \cup \Lambda_{\varphi^{-1}}.$$

Thus if $X \in \partial F_N$ is not φ -relevant and is not φ^{-1} -relevant, then the full $\hat{\iota}$ -preimage of S consists of exactly one point, namely X itself. Then $\deg(S) = 1$ and the conclusion of Theorem 7.1 obviously holds.

Recall that by Proposition 6.4 we have $\Lambda_\varphi = L(T_-(\varphi))$ and $\Lambda_{\varphi^{-1}} = L(T_+(\varphi))$. Proposition 3.12 implies that a point $X \in \partial F_N$ cannot be both φ -relevant and φ^{-1} -relevant.

Suppose now that $X \in \partial F_N$ is φ -relevant or φ^{-1} -relevant. Without loss of generality we may assume that X is φ -relevant but not φ^{-1} -relevant. Denote

$$[X] := \{X\} \cup \{Y \in \partial F_N : Y \neq X, (X, Y) \in L(T_-(\varphi))\}.$$

Then (!) together with the fact that $\Lambda_\varphi = L(T_-(\varphi))$ imply that the full $\hat{\iota}$ -preimage of S is exactly $[X]$. Thus to prove that $\deg(S) \leq 2N$ it remains to show that $\#[X] \leq 2N$. Let $T = T_-(\varphi)$. Thus T has dense F_N -orbits and the action of F_N on T is free.

Consider the map $\mathcal{Q} : \partial F_N \rightarrow \hat{T}$. Put $P = \mathcal{Q}(X)$. Part (2) of Proposition 4.11 implies that $\text{ind}_{\mathcal{Q}}(P) \leq 2N - 2$. Also, Proposition 4.1 implies that $[X] = \mathcal{Q}^{-1}(P)$, and, moreover, that if $P \in \partial T$ then $\#[X] = \#\mathcal{Q}^{-1}(P) = 1$. Since $1 \leq 2N$, the claim $\#[X] \leq 2N$ holds in this case. Thus we may assume that $P \in \overline{T}$.

Corollary 4.4 implies that $\text{ind}_{\mathcal{Q}}(P) = \#(\mathcal{Q}^{-1}(P)) - 2$. If $\#(\mathcal{Q}^{-1}(P)) \leq 2$ then $\#[X] \leq 2 \leq 2N$, as required. Suppose now that $\#(\mathcal{Q}^{-1}(P)) \geq 3$. Then $\text{ind}_{\mathcal{Q}}(P) = \#(\mathcal{Q}^{-1}(P)) - 2 = \#[X] - 2$. Since, as noted above, $\text{ind}_{\mathcal{Q}}(P) \leq 2N - 2$, it follows that $\#[X] - 2 \leq 2N - 2$ and therefore $\#[X] \leq 2N$, as required. \square

Remark 7.2. The $2N$ bound in Theorem 7.1 is sharp. Indeed, for every $N \geq 3$ the paper [29] constructs an example of a "parageometric" hyperbolic iwip φ such that some representative $\Phi \in \text{Aut}(F_N)$ of φ has $2N$ attracting fixed points and $2N - 1$ repelling fixed points in ∂F_N . After passing to a positive power, we may further assume that φ is FR. The continuity of $\hat{\iota}$ and Lemma 8.7 below imply that for any $X \in \text{Att}(\Phi)$ the point $\hat{\iota}(X)$ is fixed by $t \in M_\Phi$ and, moreover, that $\hat{\iota}(X)$ is a local attractor for t , and therefore $\hat{\iota}(X) = t^\infty$. Thus $\hat{\iota}$ maps the set $\text{Att}(\Phi)$ (which by assumption has cardinality $2N$) to a single point $S \in \partial M_\Phi$, namely, $S = t^\infty$. Since by Theorem 7.1, the $\hat{\iota}$ -preimage of S has size $\leq 2N$, it follows that $(\hat{\iota})^{-1}(S) = \text{Att}(\Phi)$ is a set of cardinality $2N$. Moreover, Lemma 4.3 of [15] implies that all elements of $\text{Att}(\Phi)$ are φ^{-1} -relevant (in the terminology of the proof of Theorem 7.1) and that $\text{Att}(\Phi)$ gives a fiber of $\mathcal{Q}_- : \partial F_N \rightarrow \hat{T}_-$.

of size $2N$, that is, there is a point $P \in \overline{T_-}$ with $(Q_-)^{-1}(P) = \text{Att}(\Phi)$. Hence $\text{ind}_Q(P) = \text{ind}_Q(T_-) = 2N - 2$.

One can also show that in this case the point $t^{-\infty} \in \partial M_\Phi$ has the $\hat{\iota}$ -preimage of size exactly $2N - 1$ (namely the set $\text{Att}(\Phi^{-1})$). Thus $\deg(t^\infty) + \deg(t^{-\infty}) = 4N - 1$ in this case and hence the $4N - 1$ bound in Theorem 8.12 below is also sharp.

8. ANALYZING THE TYPES OF FIBERS OF THE CANNON-THURSTON MAP

In this section we establish more detailed versions of Theorem B and Theorem C from the Introduction. Thus Theorem 8.9, Lemma 8.11 and Corollary 8.10 imply Theorem B.

Corollary 8.8 implies part (1) of Theorem C. Theorem 8.12 implies part (2) of Theorem C. Lemma 8.6 is part (3) of Theorem C. Proposition 8.13 and Proposition 8.14 imply parts (4) and (5) of Theorem C respectively.

Convention 8.1. For the remainder of Section 8, unless specified otherwise, we make the following assumptions and will use the following notations:

Let $\Phi \in \text{Aut}(F_N)$ be an hyperbolic iwip and let

$$(\clubsuit) \quad M_\Phi = F_N \rtimes_\Phi \langle t \rangle = \langle F_N, t | tht^{-1} = \Phi(h), h \in F_N \rangle.$$

be the mapping torus group of Φ . Let $\varphi \in \text{Out}(F_N)$ be the outer automorphism class of Φ .

Also, unless specified otherwise, we denote $G = M_\Phi$.

Remark 8.2. As noted in Remark 3.10, raising φ to a positive power does not change the trees $T_\pm(\varphi)$ and the laminations $L(T_\pm(\varphi))$, and hence, in view of Proposition 4.1, the Q -maps $\partial F_N \rightarrow \widehat{T_\pm(\varphi)}$ remain the same as well.

For any $n \geq 1$ the subgroup $D = \langle F_N, t^n \rangle \leq M_\Phi$ has index n in M_Φ and, moreover, D is naturally isomorphic to M_{Φ^n} . Thus $\partial D = \partial M_\Phi = \partial M_{\Phi^n}$ and the Cannon-Thurston maps $\partial F_N \rightarrow \partial M_\Phi$, $\partial F_N \rightarrow \partial M_{\Phi^n}$ are in fact the same. Thus we may replace Φ by its positive power without affecting any of the relevant objects (the trees $T_\pm(\varphi)$, their dual laminations, the boundary of the mapping torus group and the Cannon-Thurston map). By Proposition 3.3 in [15] any element of $\text{Out}(F_N)$ admits a positive power which is FR. Thus after possibly replacing an hyperbolic iwip φ by its positive power, for most statements in this section we may assume, if necessary, that φ is FR.

Remark 8.3. Note that if $x \in F_N$ and $t_1 = xt \in M_\Phi$ then $M_\Phi = \langle F_N, t_1 \rangle$ and for any $h \in F_N$ we have $t_1 h t_1^{-1} = x^{-1} \Phi(h) x$. Hence on the generators t_1, F_N the group M_Φ has the presentation

$$\langle F_N, t | tht^{-1} = \Phi_1(h), h \in F_N \rangle$$

where $\Phi_1(h) = x^{-1} \Phi(h) x$ for all $h \in F_N$. Thus M_{Φ_1} is naturally identified with M_Φ . Under this identification we have $\partial M_{\Phi_1} = \partial M_\Phi$, the inclusions of F_N in these groups are equal, and so the corresponding Cannon-Thurston maps are equal as functions as well.

Note that since $F_N \leq G$ is normal and infinite, the Cannon-Thurston map $\hat{\iota}: \partial F_N \rightarrow \partial G$ is surjective, since the image of $\hat{\iota}$ is the limit set of F_N in ∂G .

Definition 8.4 (Degree). For a point $S \in \partial G$ the *degree* $\deg(S)$ of S is the cardinality of the full preimage of S under $\hat{\iota}: \partial F_N \rightarrow \partial M_\Phi$. Thus we always have $1 \leq \deg(S) \leq 2N$.

Definition 8.5 (Types of points in ∂M_Φ). Let $S \in \partial M_\Phi$.

We say that:

- the point S is *regular* if $\deg(S) = 1$;
- the point S is *singular* if $\deg(S) \geq 3$;
- the point S is *semi-singular* if $\deg(S) = 2$.

In view of Corollary 6.5 and of Proposition 3.12, if S is not regular then exactly one of the following happens: for every two distinct $\hat{\iota}$ -preimages $X, Y \in \partial F_N$ of S we have $(X, Y) \in L(T_+)$; or for every two distinct $\hat{\iota}$ -preimages $X, Y \in \partial F_N$ of S we have $(X, Y) \in L(T_-)$. In the former case we say that S is of φ -type and in the latter case we say that S is of φ^{-1} -type.

It is not hard to see, from Corollary 6.5, that if $S \in \partial M_\Phi$, then for any $x \in F_N$ we have $\deg(S) = \deg(xS)$. Similarly, if S is non-regular of φ -type (respectively of φ^{-1} -type) then so is the point xS , where $x \in F_N$ is arbitrary. Moreover, Corollary 6.5 also implies that $S \in \partial M_\Phi$ is regular if and only if some (equivalently, any) $\hat{\iota}$ -preimage X of S is not an end of the laminations $L(T_\pm)$.

Lemma 8.6. *Let $x \in F_N$, $x \neq 1$. Then the rational point $x^\infty \in \partial M_\Phi$ is regular. (Here by x^∞ we mean $\lim_{n \rightarrow \infty} x^n \in \partial M_\Phi$ in the topology of $M_\Phi \cup \partial M_\Phi$).*

Proof. Put $X = \lim_{n \rightarrow \infty} x^n \in \partial F_N$ and $S = \lim_{n \rightarrow \infty} x^n \in \partial M_\Phi$. Thus $\hat{\iota}(X) = S$. Since $X \in \partial F_N$ is rational, X is not an end of either of the laminations $L(T_\pm)$. Therefore Corollary 6.5 implies that S is regular. \square

The following is a special case of Lemma 5.4:

Lemma 8.7. *Let $X \in \partial F_N$ and let $S = \hat{\iota}(X) \in \partial M_\Phi$. Then $tS = \hat{\iota}(\Phi(X))$.*

Corollary 8.8. *For any $S \in \partial G$ and any $g \in G$ we have $\deg(S) = \deg(gS)$. Moreover if S is non-regular and $g \in G$ then S is of φ -type if and only if gS is of φ -type.*

Proof. Let $g = xt^m$ where $x \in F_N$ and $m \in \mathbb{Z}$. Since $\hat{\iota}$ is surjective, there exists $X \in \partial F_N$ such that $\hat{\iota}(X) = S$. Then Lemma 8.7 implies that $gS = \hat{\iota}(x\Phi^m(X))$. Since the laminations $L(T_\pm)$ are both F_N -invariant and Φ -invariant, the conclusion of the corollary now follows from Corollary 6.5. \square

Recall that $G = M_\Phi$ and $F_N \leq G$ act on ∂G by translations, and when referring to G -orbits and F_N -orbits of points of ∂G we mean orbits with respect to these actions.

Theorem 8.9. *Let $\Phi \in \text{Out}(F_N)$ be an hyperbolic iwip and let $\hat{\iota}: \partial F_N \rightarrow \partial M_\Phi$ be the Cannon-Thurston map.*

Then:

- (a) *Let $S \in \partial M_\Phi$ be a singular point. Then S is rational, that is, there exists $g \in M_\Phi$ such that $S = g^\infty$. Moreover if S is singular of φ^{-1} -type then we can choose g as above to be of the form xt^m where $x \in F_N$ and $m > 0$. Similarly, if S is singular of φ -type then we can choose g as above to be of the form xt^m where $x \in F_N$ and $m < 0$.*
- (b) *If $S \in \partial M_\Phi$ is singular of φ^{-1} -type (respectively of φ -type) then there is no element $g = xt^m \in M_\Phi$, where $x \in F_N$, such that $S = g^\infty$ and such that $m < 0$ (respectively, $m > 0$).*

- (c) *The number of F_N -orbits of singular points of φ -type (respectively of φ^{-1} -type) in ∂F_N is $\leq 2N - 2$. Moreover,*

$$\sum_{[S]} (\deg(S) - 2) \leq 2N - 2,$$

where the summation is taken over all F_N -orbits $[S]$ of singular points of φ -type; the same inequality holds if the summation is taken over all F_N -orbits $[S]$ of singular points of φ^{-1} -type.

Proof. As noted in Remark 8.2, replacing Φ by its positive power results in taking a subgroup of finite index in M_Φ so that the boundary of M_Φ and the Cannon-Thurston map do not change. Since Φ has a positive power which is FR, it suffices to establish the conclusion of the theorem for the case where Φ is FR. Thus will assume that $\Phi \in \text{Aut}(F_N)$ is an FR hyperbolic iwip and that $\varphi \in \text{Out}(F_N)$ is the outer automorphism class of Φ .

We first establish part (a) of the theorem. Let $S \in \partial M_\Phi$ be a singular point. Without loss of generality we may assume that S is of φ^{-1} -type. Then the full $\widehat{\iota}$ -preimage of S in ∂F_N consists of k distinct points X_1, \dots, X_k where $3 \leq k \leq 2N$ and where for all $1 \leq j, r \leq k$ we have $(X_j, X_r) \in L(T_-)$. Thus the points X_1, \dots, X_k are mapped to the same point $P \in \widehat{T}_-$ under the map $\mathcal{Q} : \partial F_N \rightarrow \widehat{T}_-$ and indeed the set $\{X_1, \dots, X_k\}$ is exactly $\mathcal{Q}^{-1}(P)$. Thus $\text{ind}_{\mathcal{Q}}(P) = k - 2 \leq \text{ind}_{\mathcal{Q}}(T_-)$. Proposition 4.11 implies that $\text{ind}_{\mathcal{Q}}(T_-) = 2\text{ind}(\varphi)$. Moreover, Lemma 4.3 of [15] (which needs the assumption that φ is FR) now implies that there exists $\Phi_1 \in \text{Aut}(F_N)$ representing φ such that $\mathcal{Q}^{-1}(P)$ is exactly the set $\text{Att}(\Phi_1)$ of local attractors of Φ_1 in ∂F_N . We have $\Phi_1 = i_x \circ \Phi$ for some $x \in F_N$.

We claim that for $g = xt \in M_\Phi$ we have $gS = S$.

Recall that $X_1 \in \partial F_N$ is a local Φ_1 -attractor. Choose a rational point $X \neq X_1 \in \partial F_N$ in a Φ_1 -attracting neighborhood of X_1 . Thus $\Phi_1^n(X)$ is not an end of $L(T_\pm)$ for any $n \in \mathbb{Z}$ and $\lim_{n \rightarrow \infty} \Phi_1^n(X) = X_1$. By the choice of X the points $S_n := \widehat{\iota}(\Phi_1^n X)$ are distinct for $n = 1, 2, \dots$. By continuity of $\widehat{\iota}$ we have $\lim_{n \rightarrow \infty} S_n = \widehat{\iota}(X_1) = S$. Lemma 8.7 implies that $xtS_n = S_{n+1}$. The fact that $\lim_{n \rightarrow \infty} S_n = S$ now implies that $xtS = S$, as claimed. Since M_Φ is a torsion-free word-hyperbolic group, it now follows that S is a rational point in ∂M_Φ and moreover either $S = (xt)^\infty$ or $S = (xt)^{-\infty}$ in the topology of $M_\Phi \cup \partial M_\Phi$.

We claim that in fact $S = (xt)^\infty$. Indeed, we have a sequence of distinct points $S_n \in \partial M_\Phi$ converging to S such that $xtS_n = S_{n+1}$. Thus $\lim_{n \rightarrow \infty} (xt)^n S_1 = S$, so that S is an attracting point for the action of xt on ∂F_N . This proves part (a) of the theorem.

We will now establish part (b).

Suppose that $S \in \partial M_\Phi$ is singular of φ^{-1} -type and that there exists an element $g = xt^m \in M_\Phi$, where $x \in F_N$, such that $S = g^\infty$ and such that $m < 0$. Part (a) implies that there exists an element $g_1 = yt \in M_\Phi$, where $y \in F_N$, such that $S = g_1^\infty$ in ∂F_N . Thus $S = (xt^m)^\infty = (yt)^\infty$ where $m < 0$. Since M_Φ is torsion-free word-hyperbolic, the condition $(xt^m)^\infty = (yt)^\infty$ implies that some positive powers of xt^m and of yt are equal in M_Φ . This is impossible since $m < 0$. Thus we have established part (b) of the theorem.

We next prove part (c) of the theorem. Suppose $S_1, \dots, S_r \in \partial M_\Phi$ are singular points of φ^{-1} -type which represent distinct F_N -orbits. We claim that $r \leq 2N - 2$.

For each S_j , $j = 1, \dots, r$ construct the points $P_1, \dots, P_r \in \overline{T_-}$ as we did in the proof of part (a). Thus for each P_j the full \mathcal{Q} -preimage U_j of P_j in ∂F_N is a set of cardinality k_j satisfying $3 \leq k_j \leq 2N$ such that U_j is also the full $\widehat{\iota}$ -preimage of S_i . We claim that the points P_1, \dots, P_r are in distinct F_N -orbits in $\overline{T_-}$. Indeed, suppose not. Then after renumbering, we may assume that $xP_1 = P_2$ for some $x \in F_N$. The definition of the \mathcal{Q} now implies that $xU_1 = U_2$. Since $\widehat{\iota}(U_1) = S_1$ and $\widehat{\iota}(U_2) = S_2$, by F_N -invariance of the map $\widehat{\iota}$, we conclude that $xS_1 = S_2$. This contradicts our assumptions about S_1, \dots, S_r being in distinct F_N -orbits. Thus indeed P_1, \dots, P_r are in distinct F_N -orbits in $\overline{T_-}$. By construction we have $\text{ind}_{\mathcal{Q}}(P_j) \geq 1$ for $j = 1, \dots, r$.

We then have

$$(\diamond) \quad r \leq \sum_{j=1}^r \text{ind}_{\mathcal{Q}}(P_j) \leq \text{ind}_{\mathcal{Q}} T_- \leq 2N - 2$$

as required. Moreover, $\deg(S_j) = k_j \geq 3$ and $\text{ind}_{\mathcal{Q}}(P_j) = k_j - 2 = \deg(S_j) - 2$, and hence

$$\sum_{j=1}^r (\deg(S_j) - 2) \leq 2N - 2.$$

This establishes part (c) of the theorem. □

Corollary 8.10. *We have*

$$\sum_{[S]} (\deg(S) - 2) \leq 4N - 5$$

where the sum is taken over all F_N -orbits of singular points in ∂M_{Φ} . Moreover, the number of F_N -orbits of singular points in ∂M_{Φ} is $\leq 4N - 5$.

Proof. Part (c) of Theorem 8.9 immediately implies that $\sum_{[S]} (\deg(S) - 2) \leq 4N - 4$. Suppose that this sum is equal to $4N - 4$. Then

$$\sum (\deg(S) - 2) = \sum^I (\deg(S) - 2) = 2N - 2$$

where the first sum is taken over all F_N -orbits of singular points of φ -type and the second sum is taken over all F_N -orbits of singular points of φ^{-1} -type. Then the inequality (\diamond) in the proof part (c) of Theorem 8.9 above implies that $\text{ind}_{\mathcal{Q}} T_- = \text{ind}_{\mathcal{Q}} T_+ = 2N - 2$. Theorem 5.2 of [15] then implies that φ is of surface type, that is, comes from a pseudo-anosov homeomorphism of a compact surface with one boundary component. However, this is impossible since by assumption φ is hyperbolic.

Similarly, Theorem 8.9 implies that the number of F_N -orbits of singular points in ∂M_{Φ} is $\leq 4N - 4$. Suppose this number is actually equal to $4N - 4$. Then, by Theorem 8.9, there are $2N - 2$ F_N -orbits of singular points of φ -type and there are $2N - 2$ F_N -orbits of singular points of φ^{-1} -type. Hence, as above, the inequality (\diamond) in the proof part (c) of Theorem 8.9 implies that $\text{ind}_{\mathcal{Q}} T_- = \text{ind}_{\mathcal{Q}} T_+ = 2N - 2$, which again leads us to a contradiction. □

Note that for any $g \in G = M_{\Phi}$ the left translation action by g on ∂G coincides with the action on ∂G of the inner automorphism i_g of G , where $i_g(g') =$

$gg'g^{-1}, g' \in G$. In particular, this is true if $g \in F_N$. If $S \in \partial G$ is a rational point, then, since G is torsion-free word-hyperbolic, S can be uniquely represented as $S = g_0^\infty$ where $g_0 \in G$ is a nontrivial element which is not a proper power. Thus Corollary 8.10 implies, in particular, that for every singular $S \in \partial M_\Phi$ there exists a unique $g_0 \in M_\Phi$ such that g_0 is not a proper power and such that $g_0^\infty = S$; and, moreover, there are $\leq 4N - 5$ conjugacy classes of such elements $g_0 \in G$ and in fact there are $\leq 4N - 5$ equivalence classes of such g_0 with respect to the conjugation action by F_N on G .

Lemma 8.11. *There exists at least one singular point of φ -type and at least one singular point of φ^{-1} -type in ∂M_Φ . Hence, since by Corollary 8.8 the $\varphi^{\pm 1}$ -type of a singular point is preserved by G -translations, there exist at least 2 distinct G -orbits of singular points in ∂M_Φ .*

Proof. Proposition 4.11 implies that $\text{ind}_Q(T_+) = \text{ind}_{\text{geom}}(T_-)$. Since T_- is not a line and has some branch-points, we have $\text{ind}_{\text{geom}}(T_-) > 0$. Hence $\text{ind}_Q(T_+) > 0$ as well, so that there exists a point $P \in \overline{T_+}$ with $\text{ind}_Q(P) > 0$. Hence its preimage $Q^{-1}(P) \subseteq \partial F_N$ consists of ≥ 3 points. The $\hat{\iota}$ -image of $Q^{-1}(P)$ is a singular point $S \in \partial M_\Phi$ of φ -type.

The same argument applied to T_- yields the existence of a singular point of φ^{-1} -type. \square

Theorem 8.12. *Let $N \geq 3$ and let $\Phi \in \text{Out}(F_N)$ be an hyperbolic iwip (which is not necessarily FR) and let $\hat{\iota} : \partial F_N \rightarrow \partial M_\Phi$ be the Cannon-Thurston map. Let $g = xt^m \in M_\Phi$ where $m \neq 0$ and $x \in F_N$.*

Then

$$\deg(g^\infty) + \deg(g^{-\infty}) \leq 4N - 1.$$

Proof. Suppose, on the contrary, that $\deg(g^\infty) + \deg(g^{-\infty}) > 4N - 1$. Theorem 7.1 then implies that $\deg(g^\infty) = \deg(g^{-\infty}) = 2N$.

Thus g^∞ is a singular point. Without loss of generality we may assume that g^∞ is singular of φ^{-1} -type (the case where it is of φ -type is symmetric). Thus the $\hat{\iota}$ -preimage of g^∞ consists of $2N$ distinct points $X_1, \dots, X_{2N} \in \partial F_N$ which all have the same Q -image, which we denote by P , in $\overline{T_-}$. Hence $\text{ind}_Q(P) = 2N - 2$ and therefore $\text{ind}_Q \overline{T_-} = 2N - 2$ as well.

Then, by Theorem 5.10 of [14] either T_- is a surface tree (coming from a pseudo-anosov homeomorphism of a compact surface with one boundary component) or T_- is a pseudo-surface tree (in the terminology of [14, 15]). The former case is impossible since φ is an hyperbolic iwip and the F_N -action on T_- is free. Therefore, by Theorem 5.2 of [15], T_+ is a Levitt-type tree and φ is "parageometric".

Since $\deg(g^{-\infty}) = 2N$, exactly the same argument applies to $g^{-\infty}$. If we assume that $g^{-\infty}$ is singular of φ -type, the above argument applied to $g^{-\infty}$ implies that T_+ is pseudo-surface which is impossible since we have established that T_+ is a Levitt-type tree. Thus $g^{-\infty}$ is singular of φ^{-1} -type. Then, by part (a) of Theorem 8.9, there exist $x_1 \in F_N$ and $p > 0$ such that $g^{-\infty} = (x_1 t^p)^\infty$. Hence $g^{-\infty} = (x_1 t^p)^\infty = (t^{-m} x^{-1})^\infty$, which is impossible by part (b) of Theorem 8.9. \square

Proposition 8.13. *Let $N \geq 3$ and let $\Phi \in \text{Aut}(F_N)$ be an hyperbolic iwip. Then there are uncountably many regular points in ∂M_Φ (since $G = M_\Phi$ is countable, it follows that there are also uncountably many G -orbits of such points).*

Proof. It is well-known (e.g. from the laminary language considerations, since for any free basis \mathcal{A} of F_N there exists a freely reduced word w over $A^{\pm 1}$ such that w does not occur as a subword of any words in the laminary languages of $L(T_{\pm})$ with respect to A), that there are uncountably many points $X \in \partial F_N$ such that X is not an end of $L(T_{\pm})$.

Hence there are uncountably many regular points in ∂M_{Φ} . \square

We say that $(X, Y) \in \partial^2 F_N$ is a *simple leaf* of $L(T_+)$ if $(X, Y) \in L(T_+)$ and whenever $Z \in \partial F_N$ is such that $(X, Z) \in L(T_+)$ then $Z = Y$. The notion of a simple leaf of $L(T_-)$ is defined similarly.

Proposition 8.14. *The following hold:*

- (1) *Let $X \in \partial F_N$ and $S = \widehat{\iota}(X)$. Then $\deg(S) = 2$ if and only if there exists $Y \in \partial F_N$ such that (X, Y) is a simple leaf of $L(T_+)$ or of $L(T_-)$ (and these two possibilities are mutually exclusive).*
- (2) *The set of points of degree-2 in ∂M_{Φ} is uncountable; since G is countable, this also means that there are uncountably many G -orbits of points of degree-2 in ∂M_{Φ} .*

Proof. Part (1) follows from Corollary 6.5 and from the fact that $L(T_+)$ and $L(T_-)$ do not have any common ends.

For part (2), note that the laminations $L(T_+)$ and $L(T_-)$ are uncountable. In fact, it is the case that their sublaminations $L_{BFH}(\varphi^{\pm 1})$ are uncountable. It is also known, and can be seen, for example, from the main results of [35] giving explicit description of non-simple leaves of $L(T_+)$ and $L(T_-)$ that each of these laminations has only countably many non-simple leaves. Alternatively, the same conclusion follows from the fact that there are only countably many singular points in ∂M_{Φ} (since all such points are rational and $G = M_{\Phi}$ is countable) and that each singular point S has $\leq 2N$ ends of $L(T_{\pm})$ that are mapped to S by $\widehat{\iota}$.

Therefore each of $L(T_+)$ and $L(T_-)$ has uncountably many simple leaves. Part (2) of the lemma now follows from part (1). \square

We conclude this section with the following observation which we find noteworthy:

Remark 8.15. There exist natural F_N -equivariant surjective maps $\tau_+ : \widehat{T_+} \rightarrow \partial M_{\Phi}$ and $\tau_- : \widehat{T_-} \rightarrow \partial M_{\Phi}$ that behave well with respect to $\widehat{\iota}$ and the corresponding \mathcal{Q} -maps. Namely, for any $P \in \widehat{T_+}$ choose any $X \in \mathcal{Q}^{-1}(P)$ (for the map $\mathcal{Q} : \partial F_N \rightarrow \widehat{T_+}$) and put $\tau_+(P) := \widehat{\iota}(X)$. Corollary 6.5 implies that τ_+ is well-defined, and it is F_N -equivariant (for the original F_N -action on $\widehat{T_+}$ and the F_N -translation action on ∂F_N) by construction. The map τ_- is defined similarly.

9. OPEN PROBLEMS

Problem 9.1. Let $N \geq 3$ and let $\Phi \in \text{Aut}(F_N)$ be an arbitrary hyperbolic automorphism (not necessarily an iwip). Let $\widehat{\iota} : \partial F_N \rightarrow \partial M_{\Phi}$ be the Cannon-Thurston map (which is guaranteed to exist by Mitra's result [47]).

Is it true that $\widehat{\iota}$ is finite-to-one? Moreover, is it true that there is a uniform upper bound (depending on N but not on Φ) on the size of the fibers of $\widehat{\iota}$? Are the

singular points in ∂M_Φ necessarily rational? Is the number of F_N -orbits of singular points finite? If yes, is it bounded by a constant depending only on N ?

Problem 9.2. In [33] we proved that if $\varphi, \psi \in \text{Out}(F_N)$ are hyperbolic iwips such that $\langle \varphi, \psi \rangle \leq \text{Out}(F_N)$ is not virtually cyclic then for sufficiently large $M \geq 1$ the group $H = \langle \varphi^M, \psi^M \rangle$ is free of rank two and every nontrivial element of H is again a hyperbolic iwip. Moreover, as proved in [33] (and also follows from the results of [6]), if $\Phi, \Psi \in \text{Aut}(F_N)$ are representatives of φ, ψ then the group

$$G_M = \langle F_N, t, s | tht^{-1} = \Phi^M(h), shs^{-1} = \Psi^M(h), h \in F_N \rangle = F_N \rtimes F(t, s)$$

is word-hyperbolic. Thus the result of [47] applies here and therefore there exists a Cannon-Thurston map $\hat{\iota} : \partial F_N \rightarrow \partial G_M$.

Is it true that $\hat{\iota}$ is finite-to-one in this case? Is it true that the fibers of $\hat{\iota}$ have size $\leq 2N$?

Mitra's description [46] of the fibers of $\hat{\iota}$ in terms of "ending laminations" Λ_z , $z \in \partial F(t, s)$ does apply to $\hat{\iota} : \partial F_N \rightarrow \partial G_M$. However, to answer the above questions it is necessary to understand Λ_z where $z \in \partial F(t, s)$ is a non-rational point.

In [49] Mitra proved the existence of the Cannon-Thurston map for inclusions of vertex groups in a word-hyperbolic group G where G splits as the fundamental group of a finite graph of groups with hyperbolic vertex and edge groups such that edge-monomorphisms are quasi-isometric embeddings. This applies to the general case of the Bestvina-Feighn Combination Theorem [4]. However, unlike for the case of short exact sequences of hyperbolic groups, no description of the fibers of the Cannon-Thurston map is known for this general graph of groups setting. The result of [49] applies, for example, to mapping tori of many injective endomorphisms $\Phi : F_N \rightarrow F_N$ that are not surjective.

Problem 9.3. Let $N \geq 2$ and $\Phi : F_N \rightarrow F_N$ be an injective endomorphism with $\Phi(F_N) \lneq F_N$ being a proper malnormal subgroup, and such that Φ is irreducible in the sense of [53]. Then the mapping torus group $M_\Phi = \langle F_N, t | tht^{-1} = \Phi(h), h \in F_N \rangle$ is word-hyperbolic and, by the result of [49], there exists a Cannon-Thurston map $\hat{\iota} : \partial F_N \rightarrow \partial M_\Phi$. Is this map finite-to-one? Are the sizes of the fibers of $\hat{\iota}$ bounded by some constant depending only on N ?

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